Horn inequalities on a quiver with an involution

A refinement of Derksen-Weyman's theorem

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Quiver representations

Q $\quad x \xrightarrow{a} y \xleftarrow{b} z$ $\quad$ a finite oriented graph, $Q_0 = \{x, y, z\}$

v $\quad \mathbb{C}^{\alpha_x} \xrightarrow[v_a]{} \mathbb{C}^{\alpha_y} \xleftarrow[v_b]{} \mathbb{C}^{\alpha_z}$ $\quad \in \mathrm{Rep}_\alpha(Q) = \mathrm{Mat}_{\alpha_y,\alpha_x}(\mathbb{C}) \oplus \mathrm{Mat}_{\alpha_z,\alpha_y}(\mathbb{C})$

g $\quad g_x \qquad g_y \qquad g_z$ $\quad \in \mathrm{GL}_\alpha = \mathrm{GL}_{\alpha_x}(\mathbb{C}) \times \mathrm{GL}_{\alpha_y}(\mathbb{C}) \times \mathrm{GL}_{\alpha_z}(\mathbb{C})$

$g \cdot v$ $\quad \mathbb{C}^{\alpha_x} \xrightarrow[g_y v_a g_x^{-1}]{} \mathbb{C}^{\alpha_y} \xleftarrow[g_y v_b g_z^{-1}]{} \mathbb{C}^{\alpha_z}$ $\quad \in \mathrm{Rep}_\alpha(Q)$

Semi-invariants

From now on, Q and α are fixed. The group GL_α acts on $\mathrm{Rep}_\alpha(Q)$ hence on $\mathbb{C}[\mathrm{Rep}_\alpha(Q)]$:

$$g \cdot P : v \mapsto P(g^{-1} \cdot v)$$

To a list $\sigma \in \mathbb{Z}^{Q_0}$ we associate a multiplicative character

$$\chi_\sigma : g \mapsto \prod_{x \in Q_0} \det(g_x)^{\sigma(x)}$$

The monoid of weights

The set

$$\Sigma(Q,\alpha)=\left\{\sigma\in\mathbb{Z}^{Q_0}|\exists P\neq 0,\forall g\in \mathrm{GL}_\alpha, g\cdot P=\chi_\sigma(g)P\right\}.$$

is a monoid for $+$.

Example

If $Q=\bullet\to\bullet$ then

$$\Sigma(Q,(n,n))=\{(k,-k);k\in\mathbb{N}\}.$$

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The main theorem

Notation : $\sigma \cdot \beta = \sum_x \sigma(x)\beta(x)$.

Theorem (Derksen-Weyman, 2000)

There exists a partial order $\hookrightarrow$ on $\mathbb{N}^{Q_0}$ such that

$$\Sigma(Q,\alpha) = \left\{\sigma \in \mathbb{Z}^{Q_0} | \sigma \cdot \alpha = 0 \text{ and } \sigma \cdot \beta \leqslant 0 \text{ for all } \beta \hookrightarrow \alpha\right\}$$

Corollary

The set $\Sigma(Q,\alpha)$ is saturated : for all $n > 0$, $\sigma \in \Sigma(Q,\alpha) \Leftrightarrow n\sigma \in \Sigma(Q,\alpha)$.

Theorem (Knutson-Tao, 1999)

The Littlewood-Richardson coefficients are saturated : for all $n > 0$ and $\lambda, \mu, \nu \in \mathbb{Z}^k_{\geqslant}$,

$$\underbrace{V(\nu)}_{\text{irreducible representation of } \mathrm{GL}_k(\mathbb{C})} \subset V(\lambda) \otimes V(\mu) \Leftrightarrow V(n\nu) \subset V(n\lambda) \otimes V(n\mu).$$

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A refinement without an involution

Proposition (Horn type inequalities)

There exists a group automorphism $\beta \mapsto \tilde{\beta}$ of $\mathbb{Z}^{Q_0}$ such that the inequalities of Derksen-Weyman's theorem can be restricted to

$$\sigma \cdot \beta \leqslant 0$$

for all dimensions $\beta \neq 0$ satisfying $\tilde{\beta} \in \Sigma(Q, \alpha - \beta)$.

The semi-group $\Sigma(Q, \alpha)$ is described by induction.

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Derksen-Weyman's theorem

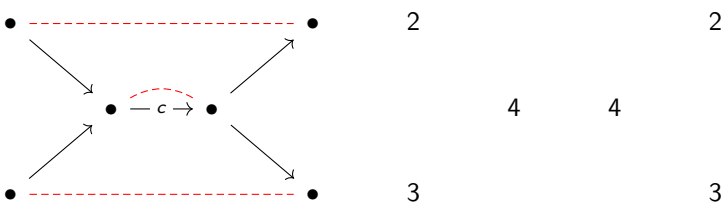
A refinement for quivers equipped with an involution

Quiver with an involution

Definition

An involution τ is an involution on the vertices and on the arrows that reverses the direction of the arrows.

It yields an involution on $\mathbb{Z}^{Q_0}$: $\tau \cdot \sigma = (\sigma_{\tau x})_x$.



A refinement with an involution

Let τ be an involution and α an invariant dimension : $\alpha = \tau \cdot \alpha$.

Lemma

The set $\Sigma(Q, \alpha)$ is stable by $-\tau$.

We can restrict the number of inequalities used to describe the anti-invariant elements.

Theorem (arXiv Horn inequalities on a quiver with an involution, 2026)

If $\sigma = -\tau \cdot \sigma$, the inequalities of Derksen-Weyman's theorem can be restricted to

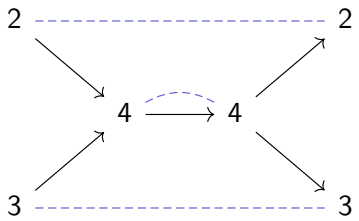
$$\sigma \cdot \beta \leqslant 0$$

for all dimensions $\beta \neq 0$ satisfying $\alpha = \beta + \gamma + \tau \cdot \beta$ and $\tilde{\beta} \in \Sigma(Q, \gamma) \cap \Sigma(Q, \tau \cdot \beta)$.

An example

Theorem (Schofield, 1992)

There exists an algorithm to compute all the β given by the three theorems above.



- Derksen-Weyman's theorem : 244 equations for $\Sigma(Q, \alpha)$.
- Its first refinement : 57 equations for $\Sigma(Q, \alpha)$.
- The last theorem : 9 equations for $\Sigma(Q, \alpha)^{-\tau}$.