Tableaux Algebra

based on joint work with

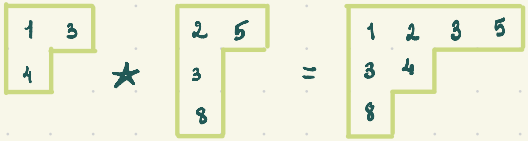
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The tableaux algebras are linear algebras / $\mathbb{C}$

$$ {}_n\mathcal{T}_m = \left\langle \text{SSYT} \begin{array}{l} \text{length} \le n \\ \text{fillings} \le m \end{array} \right\rangle $$

w/ ★ - operation obtained by concatenating and reordering rows.

eg.



Motivation:

* string semigroups associated to crystals

* toric degenerations of partial flag varieties

① Generators

$T =$  $= C_1 * \dots * C_r$

↳ Column tableaux !

② Relations

* Commutative relations $T_1 * T_2 - T_2 * T_1 \quad \forall T_i$
* Product relations $T_1 * T_2 - T_3 * T_4$ for some T_i

↳ Binomial quadratic relations

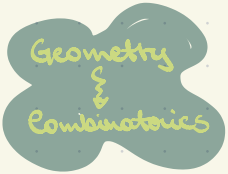
Proposition:

The tableaux algebras define projective toric varieties $\mathcal{P}_{roj}(nT_m)$

A variety V is toric if it contains a torus $(\mathbb{K}^*)^n \cong T \subseteq V$ as a open dense subset whose action extends to the whole variety V.

$$T \underset{\text{dense}}{\hookrightarrow} V \circlearrowright T$$

Toric varieties correspond to binomial ideals!



Example:

$$ {}_2\mathcal{T}_n = \mathbb{C}\left[\boxed{a}, \begin{matrix} b \\ c \end{matrix} \;\middle|\; a, b<c \right] \Big/ \begin{matrix} \begin{matrix} a \\ c \end{matrix} * b - \begin{matrix} b \\ c \end{matrix} * a \\ \begin{matrix} a \\ c \end{matrix} * \begin{matrix} b \\ d \end{matrix} - \begin{matrix} a \\ d \end{matrix} * \begin{matrix} b \\ c \end{matrix} \end{matrix} $$

for $a<b<c<d$ avoids repetition

Example:

$$ {}_2T_n = \mathbb{C}\left[\boxed{a},\ \begin{matrix} b \\ c \end{matrix} \;\middle|\; a, b<c \right] \Big/ \left\langle \begin{matrix} a \\ c \end{matrix} * \boxed{b} - \begin{matrix} b \\ c \end{matrix} * \boxed{a},\ \begin{matrix} a \\ c \end{matrix} * \begin{matrix} b \\ d \end{matrix} - \begin{matrix} a \\ d \end{matrix} * \begin{matrix} b \\ c \end{matrix} \right\rangle $$

$\mathbb{C}[x_a, y_b]$; x_a ; $x_b y_c$

for $a<b<c<d$ avoids repetition

$$ (\mathbb{C}^*)^k \subseteq \operatorname{Spec} \mathbb{C}[x_a, y_b] \xleftrightarrow{\text{DENSE}} \operatorname{Spec}({}_2T_n) \subseteq \mathbb{A}^{n+\binom{n}{2}} $$

+ TORIC ACTION - EXTENDS

Application: Partial Flag Varieties

$$Fl^n_{\underline{d}} = \{ V_1 \subseteq \dots \subseteq V_\ell \subseteq \mathbb{C}^n \mid \dim V_i = d_i \}$$

$$\hookrightarrow \mathbb{P}(\Lambda^{d_1}\mathbb{C}^n) \times \dots \times \mathbb{P}(\Lambda^{d_\ell}\mathbb{C}^n)$$

There are 2 kinds of relations

- ★ (i-th) Grassmann-Plücker relations — garantee the (i-th) component comes from a d_i-dim v. sp. V_i
- ★ Incidence relations — garantee that $V_i \subseteq V_{i+1}$ $\forall i$

Example

$$\mathcal{F}\ell^{m}_{1,2} = \{V_1 \subseteq V_2 \subseteq \mathbb{C}^m \mid \dim V_i = i\} \hookrightarrow \mathbb{C}^m \times \Lambda^2 \mathbb{C}^m$$

$$(\operatorname{span}\{v\} \subseteq \operatorname{span}\{w, u\}) \longmapsto v \times w \wedge u$$

We have

* Grassmann – Plücker relations

$$P_{ab} P_{cd} - P_{ac} P_{bd} + P_{ad} P_{bc} \qquad \forall a<b<c<d\leq m$$

* Incidence relations

$$P_a P_{bc} - P_b P_{ac} + P_c P_{ac} \qquad \forall a<b<c<d\leq m$$

Example

$$\mathcal{F}l^m_{1,2} = \{V_1 \subseteq V_2 \subseteq \mathbb{C}^m \mid \dim V_i = i\} \longleftrightarrow \mathbb{C}^m \times \Lambda^2 \mathbb{C}^m$$
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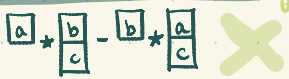
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* Incidence relations

$$P_a P_{bc} - P_b P_{ac} + P_c P_{ac} \quad \forall a<b<c<d \leq m$$



Putting it together, we have

$$A = \mathbb{C}[x_a, x_{bc}, t] \Big/ \begin{matrix} x_a x_{bc} - x_b x_{ac} + t x_c x_{ab} \\ t x_{ab} x_{cd} - x_{ac} x_{bd} + x_{ad} x_{bc} \end{matrix}$$

$\rightsquigarrow$ $\mathrm{Proj}\, A \longrightarrow \mathrm{Spec}\, \mathbb{C}[t]$

with fibers

$$\mathrm{Proj}\, A_z \cong \mathrm{Proj}\left(A/_{t-z}\right) \cong \mathcal{G}\ell^m_{1,2} \quad \forall z \neq 0$$

$$\mathrm{Proj}\, A_0 \cong \mathrm{Proj}\left(A/_{t}\right) \cong \mathrm{Proj}\left(\mathbb{C}\tau_m\right)$$

Putting it together, we have

$$A = \mathbb{C}[x_a, x_{bc}, t] \Big/ \begin{matrix} x_a x_{bc} - x_b x_{ac} + t x_c x_{ab} \\ t x_{ab} x_{cd} - x_{ac} x_{bd} + x_{ad} x_{bc} \end{matrix}$$

$\rightsquigarrow \operatorname{Proj} A \longrightarrow \operatorname{Spec} \mathbb{C}[t]$

with fibers

$$\operatorname{Proj} A_z \cong \operatorname{Proj}(A/_{t-z}) \cong \mathcal{H}^m_{1,2} \quad \forall z \neq 0$$

$$\operatorname{Proj} A_0 \cong \operatorname{Proj}(A/t) \cong \operatorname{Proj}(_2T_m)$$

Theorem: $\operatorname{Proj}(_nT_m)$ is a toric degeneration of $\mathcal{H}^m_{1,2,\dots,n}$

Based on work of Gonciulea - Lakshmibai

Theorem: $\mathrm{Proj}(n \top m)$ is a toric degeneration of $\mathcal{F}\ell^{n}_{1,2\ldots,m}$

Proposition:

The minimal # of Grassmann-Plücker relations in $Gr(i,m)$ is

$$\frac{1}{2} \sum_{k=\max\{0,2i-m\}}^{i-1} \frac{i-k-1}{i-k+1} \binom{m}{k,\ i-k,\ i-k}$$

The minimal # of incidence relations in $\mathcal{F}\ell^{m}_{1,\ldots,n}$ is

$$\sum_{i=1}^{n} \sum_{k=\max\{0,2i-m\}}^{i-1} \sum_{j=1}^{\min\{n-i,\ n-2i+k\}} \frac{i-k}{i+j-k+1} \binom{m}{k,\ i-k,\ i+j-k}$$


Thank you