UNIVERSITY OF GRAZ
Department of Mathematics and
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Magnitude of module categories

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Joint work with Erlend D. Børve[2] and Martin Kalck[1]

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Magnitude = Euler characteristic of finite category

Definition

C is a **finite category** if it consists of...

- ... a finite set of objects $X \in \mathrm{ob}(C)$
- ... a finite set of morphisms $C(X, Y)$ between objects $X, Y \in \mathrm{ob}(C)$

Example: Finite group $(H, \star)$ as finite category C_H

$$\mathrm{ob}(C_H) = \{X\},\ C_H(X, X) = \{f_g : X \to X \text{ for } g \in H\},\ f_h \circ f_g = f_{h \star g}$$

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Example: Finite group $(H, \star)$ as finite category $\mathcal{C}_H$

$$\mathrm{ob}(\mathcal{C}_H) = \{X\},\ \mathcal{C}_H(X, X) = \{f_g : X \to X \text{ for } g \in H\},\ f_h \circ f_g = f_{h \star g}$$

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$\mathcal{C}$ finite category, matrix $\mathbf{Z} := (|\mathcal{C}(X,Y)|)_{X,Y\in \mathrm{ob}(\mathcal{C})}$,
then $w, c : \mathrm{ob}(\mathcal{C}) \to \mathbb{Q}$ are **weighting** on $\mathcal{C}$ and **co-weighting** on $\mathcal{C}$ if

$$\sum_{Y\in \mathrm{ob}(\mathcal{C})} \mathbf{Z}_{X,Y} \cdot w(Y) = 1 \ \forall X \in \mathrm{ob}(\mathcal{C}), \quad \sum_{X\in \mathrm{ob}(\mathcal{C})} c(X) \cdot \mathbf{Z}_{X,Y} = 1 \ \forall Y \in \mathrm{ob}(\mathcal{C}).$$

Lemma (Leinster[1])

$\mathcal{C}$ finite category with weighting w and co-weighting c. Then

$$\sum_{Y\in \mathrm{ob}(\mathcal{C})} w(Y) = \sum_{X\in \mathrm{ob}(\mathcal{C})} c(X)$$

Proof:

$$\sum_{Y\in \mathrm{ob}(\mathcal{C})} w(Y) = \sum_{Y\in \mathrm{ob}(\mathcal{C})} \left(\overbrace{\sum_{X\in \mathrm{ob}(\mathcal{C})} c(X)\cdot \mathbf{Z}_{X,Y}}^{=1} \right) w(Y) = \sum_{X\in \mathrm{ob}(\mathcal{C})} c(X) \left(\overbrace{\sum_{Y\in \mathrm{ob}(\mathcal{C})} \mathbf{Z}_{X,Y}\cdot w(Y)}^{=1} \right) = \sum_{X\in \mathrm{ob}(\mathcal{C})} c(X)$$

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Definition (Leinster[2])

C **has magnitude** if it admits weighting w and co-weighting c.
The **magnitude** of C is $\chi(C) := \sum_{X \in \mathrm{ob}(C)} w(X)$.

Example: Finite group $(H, \star)$ as finite category C_H

$$\mathrm{ob}(C_H) = \{X\} \Rightarrow 1 = \sum_{Y \in \mathrm{ob}(C)} \mathbf{Z}_{X,Y} \cdot w(Y) = \mathbf{Z}_{X,X} \cdot w(X)$$

$$\Rightarrow 1/\mathbf{Z}_{X,X} = w(X) = \chi(C_H)$$

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Magnitude of finite-dimensional algebras

The indecomposable projectives $\mathrm{ind}(A\text{-proj})$ of a finite-dimensional associative k-algebra A form a category

Theorem (Chuang–King–Leinster[3])

Let A be a finite-dimensional algebra with finite global dimension, $(S_i)_i$ the simple A-modules, $S := \bigoplus_i S_i$, then

$$\chi(\mathrm{ind}(A\text{-proj})) = \sum_{n=0}^{\infty} (-1)^n \dim \mathrm{Ext}_A^n(S, S)$$

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Magnitude of module categories

Aim: "Replace" $\operatorname{ind}(A\text{-proj})$ with $\operatorname{ind}(A\text{-mod})$ in $\chi(\operatorname{ind}(A\text{-proj})) = \sum_{n=0}^{\infty} (-1)^n \dim \operatorname{Ext}_A^n(S,S)$

Definition

- From here on, A is **representation-finite** algebra, i.e. there are finitely many indecomposable A-modules
- The **Auslander algebra**[4] of A is $\operatorname{Aus}(A) := \operatorname{End}_A(\bigoplus_{M \in \operatorname{ind}(A\text{-mod})} M)$

It holds that $\operatorname{ind}(\operatorname{Aus}(A)\text{-proj}) \cong \operatorname{ind}(A\text{-mod})^{\mathrm{op}}$

Definition

The **magnitude of the module category** of A is

$$\chi(A\text{-mod}) := \chi(\operatorname{ind}(\operatorname{Aus}(A)\text{-proj}))$$

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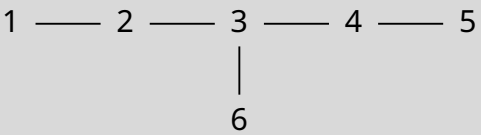
Hereditary path algebras

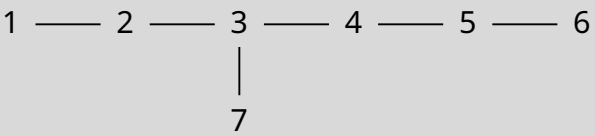
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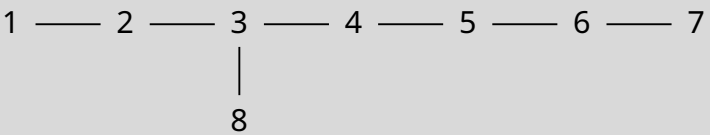
Dynkin diagram of type . . .

- A_n:

- D_n: 

- E_6: 

- E_7: 

- E_8: 

Appear in Lie theory

Hereditary path algebras

Theorem (Børve–H–Kalck)

- *Let Q be a Dynkin quiver of type Δ*
- *The **Coxeter number** of Δ is*

$$h_\Delta := 2\frac{|\textit{indecomposable } kQ\textit{-modules}|}{\operatorname{rank}(K_0(kQ\text{-mod}))}.$$

- *Then $\chi(kQ\text{-mod}) = h_\Delta - 1$. Explicitly:*

Dynkin type	A_n	D_n	E_6	E_7	E_8
$\chi(kQ\text{-mod})$	n	$2n-3$	11	17	29

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Summary

- Magnitude = Generalization of Euler characteristic
- Magnitude of finite category:

$$\chi(C) = \sum_{X \in \mathrm{ob}(C)} w(X)$$

- Magnitude of finite-dimensional algebra:

$$\chi(\mathrm{ind}(A\text{-proj})) = \sum_{n=0}^{\infty} (-1)^n \dim \mathrm{Ext}_A^n(S,S)$$

- Magnitude of module category:

$$\chi(A\text{-mod}) = \chi(\mathrm{ind}(\mathrm{Aus}(A)\text{-proj}))$$

- Q Dynkin quiver of type Δ, then

$$\chi(kQ\text{-mod}) = h_\Delta(Q) - 1$$

Thank you for your attention!