Modular Compactifications of $\mathrm{M}_{g,n}$ via Cluster Algebras

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Moduli space of curves

$\mathrm{M}_g \subset \overline{\mathrm{M}}_g$ the moduli space of **stable nodal** curves of genus g.

Can we study the birational geometry of $\overline{\mathrm{M}}_g$?

Hassett–Keel Program

Log-canonical models:

$$\overline{\mathrm{M}}_g(\alpha) := \mathrm{Proj} \bigoplus_{n \geq 0} H^0(\overline{\mathcal{M}}_g, n(K + \alpha\delta))$$

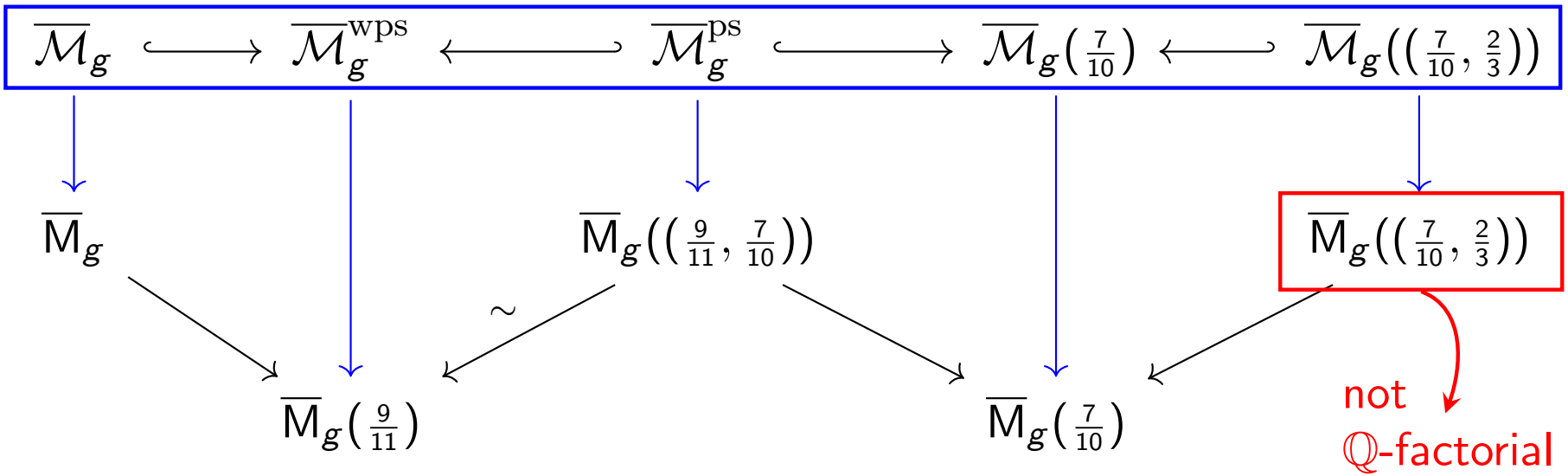
for $\alpha \in [0,1] \cap \mathbb{Q}$.

- If $\alpha > 9/11$, $K_{\overline{\mathcal{M}}_g} + \alpha\delta$ is ample;
- We expect to see a "wall-crossing picture" as α decreases.

Wall crossings of the Hassett–Keel Program

$$\overline{\mathsf{M}}_g(\alpha) := \mathsf{Proj} \bigoplus_{n \geq 0} H^0(\overline{\mathcal{M}}_g, n(K + \alpha\delta))$$

These are moduli stacks of curves with A_i sing. ($y^2 = x^{i+1}$), for $i \leq 3$.



Based on work of Alper, Fedorchuk, Hyeon, Hassett, Smyth, van der Wyck and many others

Open questions

a birational small projective morphism $X \to \overline{\mathrm{M}}_g\left(\frac{7}{10}\right)$ with X normal.

Two questions

1) Describe the partial $\mathbb{Q}$-factorializations of $\overline{\mathrm{M}}_g\left(\frac{7}{10}\right)$. Do they arise modularly? [CTV23]
2) Can we classify all possible open substacks of $\widetilde{\mathcal{M}}_g^3$ admitting a proper (or even projective) good moduli space?

The moduli stack parametrizing curves with singularities of type A_i, $i \leq 3$.

Semistability with respect to line bundles

For any $L \in \mathrm{Pic}(\overline{\mathcal{M}}_g(\frac{7}{10}))$, we can consider the open subset of L-semistable points

$$\overline{\mathcal{M}}_g\left(\frac{7}{10}\right)^{L-ss} \subset \overline{\mathcal{M}}_g\left(\frac{7}{10}\right),$$

this induces a wall-crossing decomposition of

$$\mathrm{NS}_g := \frac{\mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathcal{M}}_g(\frac{7}{10})\right)}{\mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathrm{M}}_g(\frac{7}{10})\right)} = \bigsqcup_{i\in I} \mathcal{C}_i$$

Theorem [G. 26]

Let $g > 2$. The regions $\{\mathcal{C}_i\}_{i\in I}$ form a complete rational polyhedral fan Σ_g in NS_g. Moreover, Σ_g is the cluster fan of a cluster algebra of type:

$$\begin{cases} A_1 \times A_{\frac{g-4}{2}} & \text{if } 2 \mid g, \\ A_1 \times C_{\frac{g-3}{2}} & \text{if } 2 \nmid g. \end{cases}$$

Theorem (Classification, [G. 26])

Let $\mathcal{U} \subset \widetilde{\mathcal{M}}_g^3$ be an open substack with a projective good moduli space. Then one of the following holds:

i) *$\mathcal{U} \subset \widetilde{\mathcal{M}}_g^2$, then it is one of $\overline{\mathcal{M}}_g$, $\overline{\mathcal{M}}_g^{\mathrm{ps}}$, or $\overline{\mathcal{M}}_g^{\mathrm{wps}}$;*

ii) *$\mathcal{U} \subset \overline{\mathcal{M}}_g\left(\frac{7}{10}\right)$, then $\mathcal{U} = \overline{\mathcal{M}}_g\left(\frac{7}{10}\right)^{L-\mathrm{ss}}$ for some $L \in \mathsf{NS}_g$;*

iii) *$\mathcal{U} \subset \overline{\mathcal{M}}_g^!$, the situation is analogous to ii).*

Theorem ($\mathbb{Q}$-factorializations, [G. 26])

The partial $\mathbb{Q}$-factorializations are the maps

$$\overline{\mathsf{M}}_g\left(\frac{7}{10}\right)^{L-\mathrm{ss}} \to \overline{\mathsf{M}}_g\left(\frac{7}{10}\right),$$

for $L \in \mathsf{NS}_g$. Moreover, Σ_g is the $\mathbb{Q}$-factorialization fan of $\overline{\mathsf{M}}_g\left(\frac{7}{10}\right)$.

Bonus!

Corollary (On the ample cone of $\overline{\mathsf{M}}_g$)

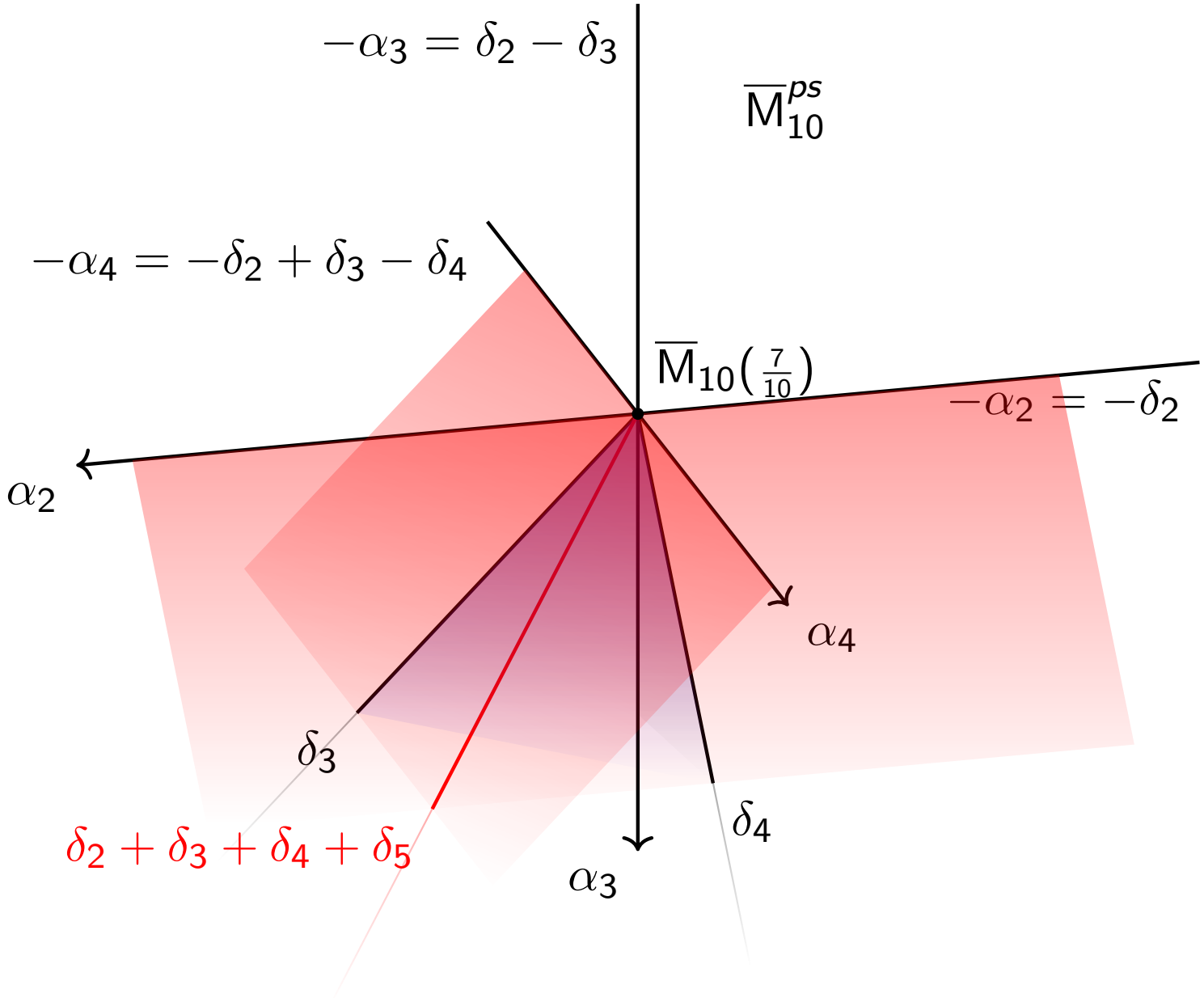
Let $g \geq 3$. The image of $\mathsf{Nef}\left(\overline{\mathsf{M}}_g\right) \cap \mathsf{Pic}_{\mathbb{Q}}\left(\overline{\mathsf{M}}_g^{\mathrm{ps}}\right)$ *under*

$$\pi\colon\ \mathsf{Pic}_{\mathbb{Q}}\left(\overline{\mathsf{M}}_g^{\mathrm{ps}}\right) \longrightarrow \frac{\mathsf{Pic}_{\mathbb{Q}}\left(\overline{\mathsf{M}}_g^{\mathrm{ps}}\right)}{\mathsf{Pic}_{\mathbb{Q}}\left(\overline{\mathsf{M}}_g(\frac{7}{10})\right)},$$

is $\overline{\mathsf{Span}_+\,\Delta_g}$*, where Δ_g is an explicit set of vectors.*

The projection π has kernel of dimension 1 if $2 \nmid g$, and 2 otherwise.

$\mathbb{Q}$-factorializations of $\overline{\mathrm{M}}_{10}(\frac{7}{10})$, cluster type $A_1 \times A_3$.



Some references I

[CTV23] Giulio Codogni, Luca Tasin, and Filippo Viviani. "On the first steps of the minimal model program for the moduli space of stable pointed curves". In: *Journal of the Institute of Mathematics of Jussieu* 22.1 (2023), pp. 145–211.

[G. 26] Davide Gori. "Alternative Compactifications of $M_{g,n}$ via Cluster Algebras and their Birational Geometry". English. https://iris.uniroma1.it/handle/11573/1759004. PhD thesis. Università La Sapienza, Roma, 2026. URL: https://iris.uniroma1.it/handle/11573/1759004.

[AHLH23] Jarod Alper, Daniel Halpern-Leistner, and Jochen Heinloth. "Existence of moduli spaces for algebraic stacks". In: *Inventiones mathematicae* 234.3 (2023), pp. 949–1038.

Some references II

[HH09] Brendan Hassett and Donghoon Hyeon. "Log canonical models for the moduli space of curves: the first divisorial contraction". In: *Transactions of the American Mathematical Society* 361.8 (2009), pp. 4471–4489.

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[AFS17a] Jarod Alper, Maksym Fedorchuk, and David Ishii Smyth. "Second flip in the Hassett–Keel program: existence of good moduli spaces". In: *Compositio Mathematica* 153.8 (2017), pp. 1584–1609.

Some references III

[AFS17b] Jarod Alper, Maksym Fedorchuk, and David Ishii Smyth. "Second flip in the Hassett–Keel program: projectivity". In: *International Mathematics Research Notices* 2017.24 (2017), pp. 7375–7419.