The FFLV basis for thin-legged covariant representations of $\mathfrak{gl}(m|n)$

joint work with Pranav Enugandla (UC Berkeley)

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Where we start: Schur–Weyl duality

Theorem (Schur, Weyl)

As a $\mathfrak{gl}_n \times S_d$-module,

$$(\mathbb{C}^n)^{\otimes d} \cong \bigoplus_{\substack{\lambda \vdash d \\ \ell(\lambda) \leq n}} V(\lambda) \otimes S^\lambda.$$

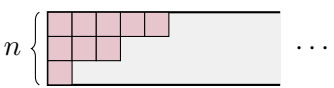
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- $V(\lambda)$: simple $\mathfrak{gl}_n$-module of highest weight λ, S^λ: Specht module of S_d.
- Combinatorially: Young diagrams with at most n rows, i.e. diagrams inside a horizontal strip of height n.



Why Lie superalgebras? A theorem of Deligne

Let $\mathcal{C}$ be a symmetric tensor category over $\mathbb{C}$.

Theorem (Deligne 2002) [Modulo technicalities]

Every symmetric tensor category over $\mathbb{C}$ of moderate growth is equivalent to $\mathrm{Rep}(G)$ for an affine supergroup scheme G.

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But what are Lie superalgebras now?

The Lie superalgebra $\mathfrak{gl}(m|n)$

$$\mathfrak{gl}(m|n) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \;\middle|\; A \in \mathbb{C}^{m\times m},\ B \in \mathbb{C}^{m\times n},\ C \in \mathbb{C}^{n\times m},\ D \in \mathbb{C}^{n\times n} \right\}$$

- $\mathbb{Z}/2\mathbb{Z}$-grading $\mathfrak{gl}(m|n) = \mathfrak{gl}(m|n)_{\bar{0}} \oplus \mathfrak{gl}(m|n)_{\bar{1}}$:
  even part = block diagonal ($B = C = 0$), $\cong \mathfrak{gl}_m \oplus \mathfrak{gl}_n$;
  odd part = block anti-diagonal ($A = D = 0$)
- bracket of homogeneous elements: $[X, Y] = XY - (-1)^{|X|\,|Y|}\, YX$

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Example: $\mathfrak{gl}(1|1)$

$E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are odd, so their bracket is an anticommutator:

$$[E_{12}, E_{21}] = E_{12}E_{21} + E_{21}E_{12} = E_{11} + E_{22} = \mathrm{Id}.$$

Also $[E_{12} + E_{21}, E_{12} + E_{21}] = 2\,\mathrm{Id} \neq 0.$

More peculiarities

- The Borel subalgebras are not conjugate under the Weyl group $S_m \times S_n$
- (Odd) roots are isotropic, i.e. $(\alpha, \alpha) = 0$
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Back on track: Schur–Weyl duality for $\mathfrak{gl}(m|n)$

$\mathbb{C}^{m|n}$: natural representation, basis $e_1, \dots, e_m$ (even), $e_{\bar{1}}, \dots, e_{\bar{n}}$ (odd); S_d acts on $(\mathbb{C}^{m|n})^{\otimes d}$ with signs.

Theorem (Sergeev 1984, Berele–Regev 1987)

As a $\mathfrak{gl}(m|n) \times S_d$-module, $(\mathbb{C}^{m|n})^{\otimes d} \cong \bigoplus_{\nu \in H(m,n;d)} V(\nu) \otimes S^\nu$, where

$H(m,n;d) = \{\nu \vdash d \ : \ \nu_{m+1} \leq n\}$ = partitions of d in the (m,n)-hook.

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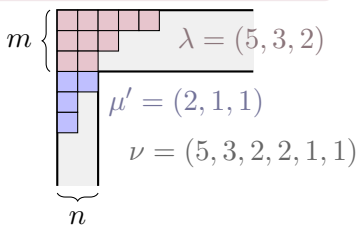
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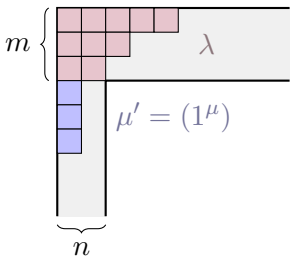
- So $(\mathbb{C}^{m|n})^{\otimes d}$ *is* semisimple!
- Its constituents are the covariant representations.
- Their Young diagrams live in the hook: rows $1, \dots, m$ arbitrary, rows $m+1, m+2, \dots$ of length $\le n$.



Thin-legged diagrams

Definition

A covariant weight is thin-legged if it is of the form $(\lambda \,|\, \mu, 0^{n-1})$ with $\mu \in \mathbb{N}_0$; equivalently $\mu' = (1^\mu)$: the leg is a single column of μ boxes hanging below row m.



PBW filtration and the FFLV polytope

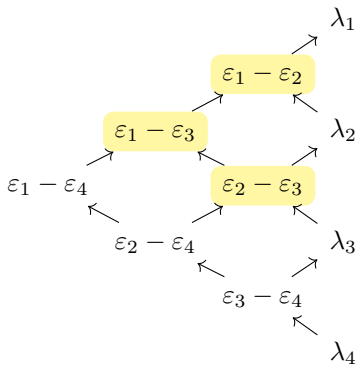
The *Gelfand-Tsetlin pattern* on the right gives rise to a basis

Theorem (Feigin–Fourier–Littelmann 2011, for $\mathfrak{gl}_m$)

$\{ f^{\mathbf{s}} v_\lambda = \prod_{i<j} f_{\varepsilon_i - \varepsilon_j}^{s_{i,j}} v_\lambda \}$ is a basis of $\operatorname{gr} V(\lambda)$, where $\mathbf{s}$ runs over the lattice points of the FFLV polytope

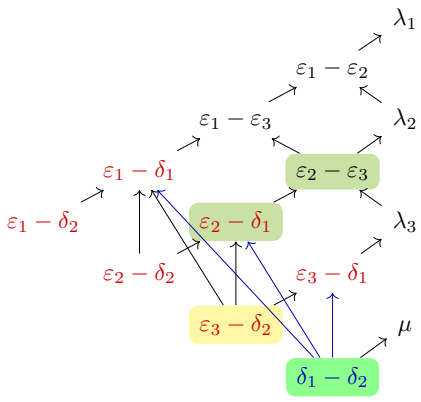
$$P(\lambda) = \Big\{ \mathbf{s} \in \mathbb{R}_{\geq 0}^{\Phi^+} \;\Big|\; \sum_{\beta \in \mathbf{p}} s_\beta \;\leq\; \lambda_i - \lambda_j \;\text{ for all } \mathbf{p} \Big\},$$

$\mathbf{p}$ running over the *Dyck paths* from $\varepsilon_i - \varepsilon_{i+1}$ to $\varepsilon_{j-1} - \varepsilon_j$.



$\mathfrak{gl}_4$: the highlighted path gives $s_{1,2} + s_{1,3} + s_{2,3} \leq \lambda_1 - \lambda_3$.

Our pattern: extended Dyck paths for $(\lambda \,|\, \mu, 0^{n-1})$



$\mathfrak{gl}(3|2)$, weight $(\lambda_1, \lambda_2, \lambda_3 \,|\, \mu, 0)$

One coordinate s_α per node; paths follow the arrows.

- **Even paths**: FFLV, $\sum_{\mathbf{p}} s_\beta \leq \lambda_i - \lambda_j$.
- **Extended paths:**
  - from an odd root up to λ_r: $\sum_{\mathbf{p}} s_\beta \leq \lambda_r$; and $s_{\varepsilon_i - \delta_j} \leq 1$, such as

$$s_{\varepsilon_3 - \delta_2} + s_{\varepsilon_2 - \delta_1} + s_{\varepsilon_2 - \varepsilon_3} \leq \lambda_2$$

  - from $s_{\delta_1 - \delta_2}$ up to λ_r:

$$\mu \cdot \sum_{t \geq 1} s_{p(t)} \;+\; s_{\delta_1 - \delta_2} \;\leq\; \mu \cdot \lambda_r$$

such as

$$\mu \cdot (s_{\varepsilon_2 - \delta_1} + s_{\varepsilon_2 - e_3}) \;+\; s_{\delta_1 - \delta_2} \;\leq\; \mu \cdot \lambda_2$$

The main result

Theorem A (A.–Enugandla)

Let λ be a partition with $\lambda_m \neq 0$ (any λ if $\mu = 0$) and $\mu \in \mathbb{N}_0$. Then $P(\lambda \,|\, \mu, 0^{n-1}) \subseteq \mathbb{R}^{d_1} \times [0,1]^{d_2}$ is a lattice polytope with

1. $\{ f^{\mathbf{s}} v_{(\lambda|\mu,0^{n-1})} \mid \mathbf{s} \in S(\lambda|\mu,0^{n-1}) = P(\lambda|\mu,0^{n-1}) \cap \mathbb{Z}^{d_1+d_2} \}$ is a basis of $\operatorname{gr} V(\lambda \,|\, \mu, 0^{n-1})$;
2. Minkowski decomposition
   - $S(\lambda \,|\, \mu, 0^{n-1}) = \big(S(\lambda - 1^m \,|\, 0^n) + S(1^m \,|\, \mu, 0^{n-1})\big) \cap \{0,1\}^{d_2}$.
   - $S(\lambda + \mu \,|\, 0^n) = \big(S(\lambda \,|\, 0^n) + S(\mu \,|\, 0^n)\big) \cap \{0,1\}^{d_2}$.

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   - $S(\lambda + \mu \,|\, 0^n) = \big(S(\lambda \,|\, 0^n) + S(\mu \,|\, 0^n)\big) \cap \{0,1\}^{d_2}$.

Application: flat proper morphism of superschemes $\kappa : \operatorname{Proj} \mathcal{R} \to \mathbb{A}^{1|0}$ with fibres

$$\kappa^{-1}(a) \;\cong\; \begin{cases} \operatorname{Proj} \mathbb{C}[G/P]_\nu \;=\; G/P \hookrightarrow \mathbb{P}V(\nu) & a \neq 0, \\ \operatorname{Proj} \mathbb{C}\big[x^{\mathbf{s}} u \mid \mathbf{s} \in S(\lambda|0^n)\big] & a = 0, \end{cases}$$

and $\operatorname{Spec} \mathbb{C}\big[x^{\mathbf{s}} u \mid \mathbf{s} \in S(\lambda|0^n)\big]$ is a toric supervariety in the sense of Jankowski.

Problems

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Thank you!