Spherical representation spaces of quivers

joint work with Markus Reineke

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CARE II - 23.09.2026

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Idea:

- Translate the problem of $B_{\mathbf{d}}$-orbits back to $G_{\mathbf{d}}$-orbits.
- Work with the roots of Q.

The leg extended quiver

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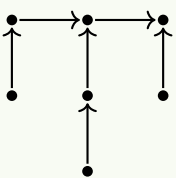
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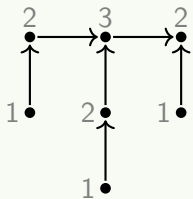
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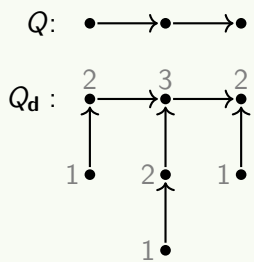
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We call a representation V of $Q_{\mathbf{d}}$ of dimension vector $\widehat{\mathbf{d}}$ of **flag type** if all "new arrows" are represented by injective maps. $\Rightarrow R^{flag}(Q_{\mathbf{d}}, \widehat{\mathbf{d}})$

Associated fibre bundles

Lemma: There exists a $G_{\widehat{\mathbf{d}}}$-equivariant isomorphism

$$R^{flag}(Q_{\mathbf{d}}, \widehat{\mathbf{d}}) \simeq G_{\widehat{\mathbf{d}}} \times^{B_{\mathbf{d}}} R(Q, \mathbf{d}).$$

In particular, $B_{\mathbf{d}}$-orbits in $R(Q, \mathbf{d})$ correspond bijectively to $G_{\widehat{\mathbf{d}}}$-orbits in $R^{flag}(Q_{\mathbf{d}}, \widehat{\mathbf{d}})$.

Imaginary roots

Thm[Kac]:

1. $\exists$ indecomposable representation of Q of dimension vector $\mathbf{d} \Leftrightarrow \mathbf{d} \in \Delta_+$
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Corollary:
$G_{\widehat{\mathbf{d}}}$ acts on $R^{flag}(Q_{\mathbf{d}}, \widehat{\mathbf{d}})$ with only finitely many orbits if and only if there exists no gentle imaginary root **e** with $\mathbf{e} \leq \widehat{\mathbf{d}}$.

Classification

Thm:

1. If Q is Dynkin, then there are no imaginary roots and

$$\Delta = \Delta^{re} = \{\alpha \in \mathbb{Z}Q_0 \,|\, (\alpha, \alpha) = 2\}.$$

2. If Q is euclidean, then

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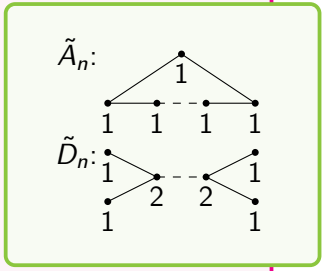
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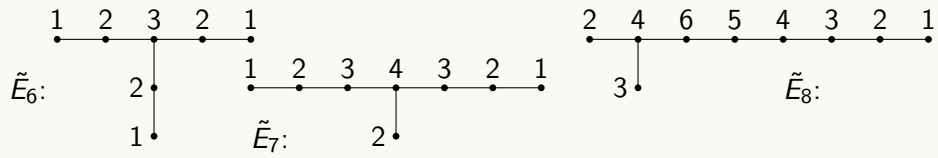
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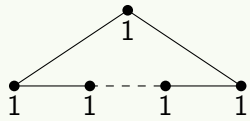
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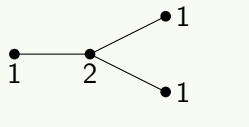




Classification - Summary

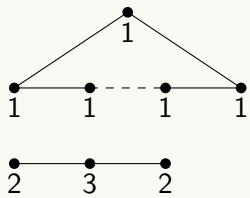
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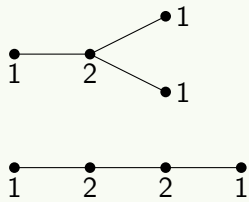




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Lemma: If $B_{\mathbf{d}}$ acts on $R(Q, \mathbf{d})$ with infinitely many orbits, then the same holds for all $\mathbf{e}$ with $\mathbf{e} \geq \mathbf{d}$.

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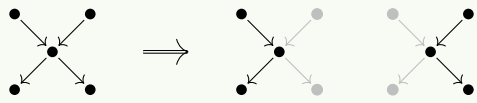
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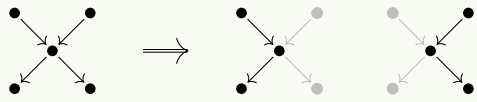


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Example:



A connected sum of quiver settings $(Q, \mathbf{d})$ is called **thin** if $d_i = 1$.

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