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Campana points on generalised diagonal forms

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Warm-up: Integer solutions in boxes

- $n, d \in \mathbb{N}$
- $P \in \mathbb{Z}[X_1, \dots, X_n]$ homogeneous of degree d (form)

$\rightsquigarrow$ Proj. variety $V = \{P = 0\} \subset \mathbb{P}^{n-1}_{\mathbb{Q}}$.

Study cardinality of

$$N(B) = \left\{ (u_1 : \cdots : u_n) \in \mathbb{P}^{n-1}(\mathbb{Q}) \;\middle|\; \begin{array}{l} |u_i| \leq B \text{ for all } i \in \{1, \dots, n\}, \\ P(u_1, \dots, u_n) = 0 \end{array} \right\}$$

as $B \to \infty$.

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$$\sigma \coloneqq \dim\{\mathbf{x} \in \mathbb{A}^n_{\mathbb{C}} : \nabla P(\mathbf{x}) = 0\}.$$

Theorem (Birch 1962)

If $n - \sigma > (d-1)2^d$ then

$$\#N(B) = cB^{n-d} + O(B^{n-d-\delta})$$

for some explicit constant c and some $\delta > 0$.

Conjecture (Manin 1989)

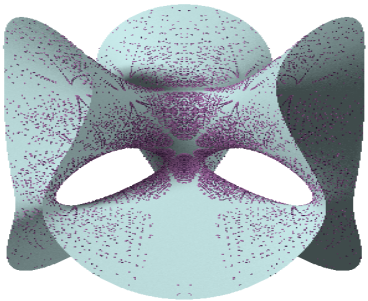
- X/k smooth projective Fano variety over a number field k
- Assume $X(k)$ Zariski dense
- $H : X(k) \to \mathbb{R}_{\geq 0}$ be a height function on X.

Then there is some nonempty Zariski-open $U \subseteq X$ such that

$$\boxed{N_U(B) \sim c_{X,H}\, B^a (\log B)^{\rho-1}} \quad \text{as } B \to \infty.$$

where $a > 0, \rho = \mathrm{rk}\,\mathrm{Pic}(X)$. The constant $c_{X,H} > 0$ is the predicted leading constant (refined by Peyre).There are counterexamples: One may also have to remove "thin sets".

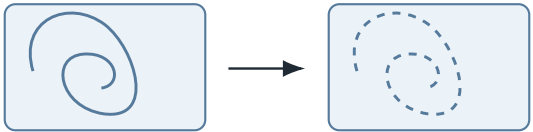
The Clebsch cubic surface in $\mathbb{R}^3$



$$0 = 81(x^3 + y^3 + z^3) - 189\big(x^2(y+z) + y^2(z+x) + z^2(x+y)\big)$$
$$+ 54xyz + 126(xy + yz + zx) - 9(x^2 + y^2 + z^2) - 9(x + y + z) + 1.$$

Visualization of bounded-height rational points on the Clebsch surface, outside the 27 lines.
Credit: U. Derenthal and O. Labs, CC BY-SA 3.0.

From rational to integral points



Rational points: easy

Integral points: hard

$$\mathbb{P}^2_{\mathbb{Q}}, \qquad [x : y : z].$$

Counting rational points of height $\leq B$ counts primitive lattice points in a box:

$$\#\{X \in \mathbb{P}^2(\mathbb{Q}) : \mathrm{H}(X) \leq B\} \sim \frac{4}{\zeta(3)} B^3.$$

no restrictions on $x^3 + y^3 + z^3$.

$$U = \mathbb{P}^2 \setminus D, \qquad D : \; x^3 + y^3 + z^3 = 0.$$

Integral points on U correspond to

$$x^3 + y^3 + z^3 = \pm 1,$$

so, after changing all signs if necessary,

$$x^3 + y^3 + z^3 = 1.$$

$$K_{\mathbb{P}^2} = -3H, \qquad D \sim 3H.$$

Hence

$$\boxed{K_{\mathbb{P}^2} + D \sim 0.}$$

So the affine open is log Calabi–Yau – No general prediction for count of integral points!

$$\boxed{\text{no restrictions on } x^3+y^3+z^3} \leftrightarrow \boxed{???} \leftrightarrow \boxed{\mathrm{v}_p(x^3+y^3+z^3) = 0 \text{ for all } p.}$$

$$m \in \mathbb{Z}_{>0},\ \boxed{???} = \boxed{\mathrm{v}_p(x^3+y^3+z^3) \in \{0\} \cup \{m, m+1, m+2, \dots\} \text{ for all } p}$$

Campana points

$k \in \mathbb{Z}, m \in \mathbb{N}, p$ prime

Definition

We say k is m-full if $p \mid k \Rightarrow p^m \mid k$ for all p.

Example

- Every integer is 1-full.
- $1000 = 2^3 \cdot 5^3$ is 2-full and 3-full.
- $72 = 2^3 \cdot 3^2$ is 2-full but not 3-full.

Counting powerful solutions

$P \in \mathbb{Z}[X_1, \dots, X_n]$,

$\mathfrak{M} := (m_1, \dots, m_n) \in \mathbb{Z}^n_{>0}$.

$$N_{\mathfrak{M}}(B) := \left\{ (u_1 : \cdots : u_n) \in \mathbb{P}^{n-1}(\mathbb{Q}) \;\middle|\; \begin{array}{l} |u_i| \leq B \text{ for all } i \in \{1, \dots, n\}, \\ P(u_1, \dots, u_n) = 0 \\ u_i \; m_i\text{-full for all } i \in \{1, \dots, n\}. \end{array} \right\}$$

Conjecture by Pieropan-Smeets-Tanimoto-Várilly-Alvarado (2019) predicts asymptotic in the spirit of Manins conjecture!

Squarefull solutions of linear forms

$\mathfrak{M} = (2, \dots, 2), P = X_1 + \cdots + X_{n-1} - X_n.$

- $N_{\mathfrak{M}}(B) \sim c \cdot B^{(n-2)/2}$ if $n \geq 5$ (Van Valckenborgh 2011)
- $N_{\mathfrak{M}}(B) \ll B^{3/5-3/1555+\varepsilon}$ if $n = 3$ (V.V.-Browning 2012, Heath-Brown 2026)

Powerful solutions of diagonal forms

$\mathfrak{M} = (m_1, \dots, m_n), m_1 \leq \cdots \leq m_n, P = c_1 X_1^d + \cdots + c_n X_n^d, c_i \in \mathbb{Z}_{\neq 0}.$

- $N_{\mathfrak{M}}(B) \sim c \cdot B^{\sum_{i=1}^n (1/m_i)-1}$ if $d = 1$ and $\sum_{i \neq n} 1/(m_i(m_i+1)) > 1$ (Browning-Yamagishi 2019)
- $N_{\mathfrak{M}}(B) \sim c \cdot B^{\sum_{i=1}^n (1/m_i)-d}$ for $d \geq 1$ and similar sum condition. (Balestrieri-Brandes-Kaesberg-Ortmann-Pieropan-Winter 2023)

$P = \sum_{l=1}^{n/d} c_l X_{(l-1)d+1} \cdots X_{ld}$. Suppose $m_1 = \min m_i$.

Case (i): $c_1 \overbrace{X_1 \cdots X_d}^{m_1=\cdots=m_d} + c_2 \overbrace{X_{d+1} \cdots X_{2d}}^{m_{d+1}=\cdots=m_{2d}} + \ldots$ (equally weighted terms)

Case (ii): $c_1 \overbrace{X_1}^{m_1} \overbrace{X_2}^{m_2} + c_2 \overbrace{X_3}^{m_3} \overbrace{X_4}^{m_4} + \ldots$ (degree 2 bihomogeneous)

Theorem (U. 2026+)

- Case (i): $N_{\mathfrak{M}}(B) \sim cB^{\sum_{i=1}^{n}(1/m_i)-d}$ if $\sum_{l\neq 1}^{n/d} \frac{1}{m_{ld}(m_{ld}+1)} > 1 + \frac{1}{m_1}$

  (+ smaller conditions)

- Case (ii): $N_{\mathfrak{M}}(B) \sim cB^{\sum_{i=1}^{n}(1/m_i)-2}$ if $\sum_{l\neq 1}^{n/d} \frac{1}{2^{m_{2l-1}+m_{2l-1}}(m_{21}-1+m_{2l}-1)} > 1 + \frac{1}{m_1}$

  (+ ”enough equal weighted terms")

$\rightsquigarrow N_{\mathfrak{M}}(B) = \sum_{\gamma_i} N_{\gamma}(B)$

$$N_{\gamma}(B) = \left\{ (h_1 : \cdots : h_n) \in \mathbb{P}^{n-1}(\mathbb{Q}) \;\middle|\; \begin{array}{l} h_1, \ldots, h_n \in \mathbb{Z}_{\neq 0}, \\ |h_i| \leq (B/\gamma_i)^{1/m_i} \text{ for all } i \in \{1, \ldots, n\}, \\ F(h_1, \ldots, h_n) = 0 \end{array} \right\}$$

Well suited for application of circle method. (Multidimensional Vinogradov mean value estimates)

Difficulties:

- Lopsided boxes of summation if the m_i are arbitrary
- Good dependence on γ_i in error terms needed