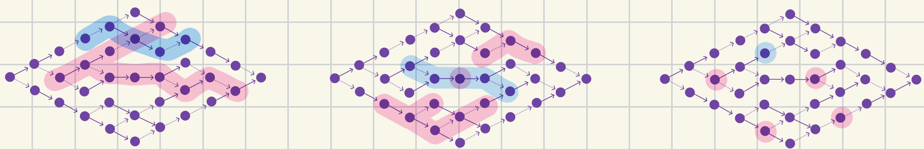
Kotska–Foulkes polynomials and atomic decomposition



Kotska–Foulkes polynomials' many interpretations: $K_{\lambda\mu}(q)$

- deformations of Kotska numbers
  - transition matrices between bases of symmetric polynomials over $\mathbb{Q}(q)$
  - q-weight multiplicities
- affine Kazhdan–Lusztig polynomials
- jump polynomials in the Brylinski filtrations

...

as q-analogues of Kotska numbers ie weight multiplicities

$$K_{\lambda,\mu} = \dim V(\lambda)_\mu$$

as q-analogues of Kostka numbers

ie weight multiplicities

$$K_{\lambda,\mu} = \dim V(\lambda)_\mu = \# \mathcal{B}(\lambda)_\mu$$

Crystal bases

eg Adjoint representation $V(\epsilon_1 - \epsilon_4)^{\mathfrak{sl}_4}$

$$\begin{tikzcd}
 & & {[e_1,[e_2,e_3]]} \arrow[dl] \arrow[dr] & & \\
 & {[e_1,e_2]} \arrow[dl] \arrow[dr] & & {[e_2,e_3]} \arrow[dl] \arrow[dr] & \\
e_1 \arrow[d] & & e_2 \arrow[d] & & e_3 \arrow[d] \\
h_1 \arrow[d] & & h_2 \arrow[d] & & h_3 \arrow[d] \\
f_1 \arrow[dr] & & f_2 \arrow[dl] \arrow[dr] & & f_3 \arrow[dl] \\
 & {[f_1,f_2]} \arrow[dr] & & {[f_2,f_3]} \arrow[dl] & \\
 & & {[f_1,[f_2,f_3]]} & &
\end{tikzcd}$$

$\mathcal{B}(\lambda)_0$

as q-analogues of Kostka numbers

ie weight multiplicities

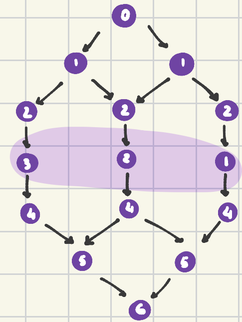
$$K_{\lambda,\mu} = \dim V(\lambda)_\mu = \# \mathcal{B}(\lambda)_\mu$$

$$\Downarrow$$

$$K_{\lambda\mu}(q) = \sum_{v \in \mathcal{B}(\lambda)_\mu} q^{ch(v)}$$

for some statistic

$$ch : \mathcal{B}(\lambda) \longrightarrow \mathbb{Z}_{\geq 0}$$



$$K_{\lambda,0} = q^3 + q^2 + q^1$$

Theorems:

A charge statistic ch st $K_{\lambda,\mu}(q) = \sum_{B(\lambda)_\mu}' q^{ch(v)}$ exists

In type A — Lascoux-Schützenberger '78

In types B, C & D for small cases $|\lambda| \leq 3$ or rk 2 $\mu = 0$ — Lecouvey '05

In type C for $\lambda = \boxed{\;|\;|\,\cdots\,|\;} = n\omega_1$ — Dołęga - Gerber - Torres '20

In type A — Patimo '21

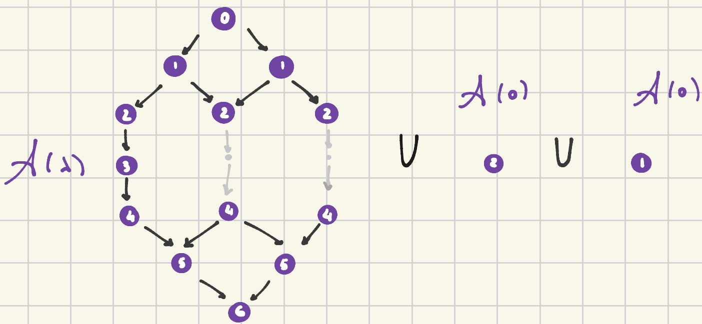
In type $C_2 = B_2$ — Patimo - Torres '23

In type C — Choi - Kim - Lee '25

Atomic decomposition of crystals

$$\mathcal{B}(\lambda) = \bigcup_i \mathcal{A}(\gamma_i)$$

where $\mathcal{A}(\gamma_i) \sim \{\leq \gamma_i\} = \left\{ \begin{matrix} \text{wts in} \\ \mathcal{B}(\gamma_i) \end{matrix} \right\}$



q-Atomic decomposition of crystals

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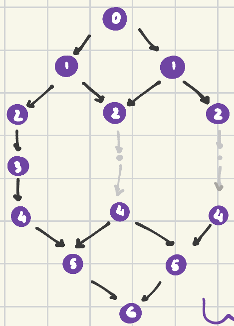
where $\mathcal{A}(\gamma_i) \sim \{\leq \gamma_i\} = \left\{ \begin{matrix} \text{wts in} \\ \mathcal{B}(\gamma_i) \end{matrix} \right\}$

$\mathcal{A}(\lambda)$



$\cup$ $\mathcal{A}(0)$ ③ $\cup$ $\mathcal{A}(0)$ ①

$\leadsto$ $A_{\lambda\lambda} = 1$

$A_{\lambda 0} = q^2 + q$

BETTER

$$K_{\lambda\mu} = \sum_{\mu \leq \gamma \leq \lambda} q^{\langle \gamma - \mu, \rho^\vee \rangle} A_{\lambda\gamma}$$

Theorems:

In type A, every crystal admits q-atomic decomposition

[Lascoux '91 & Shimozono '01]

In types B, C & D, the crystal $\mathcal{B}_\lambda$ admits atomic decomposition for n big enough (depends on λ & type)

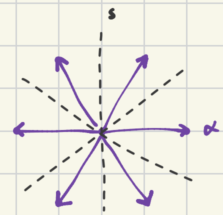
[Lecouvey-Lenart '19]

In type G_2, every crystal admits q-atomic decomposition

as affine Kazhdan – Lusztig polynomials

Weyl groups

Finite $W_f \curvearrowright \Phi$ roots



as affine Kazhdan-Lusztig polynomials

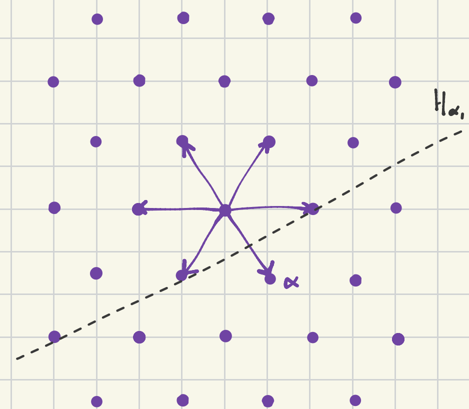
Weyl groups

Finite $W_f \curvearrowright \Phi$ roots

Affine $W_a \curvearrowright \mathbb{Z}\Phi$ root lattice

$$W_a = W_f \ltimes \mathbb{Z}\Phi$$

$\rightsquigarrow$ by translations



as affine Kazhdan – Lusztig polynomials

Weyl groups

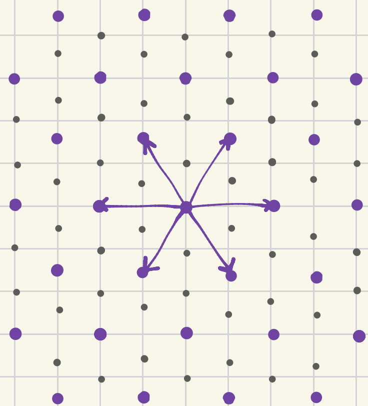
Finite $W_f \curvearrowright \Phi$ roots

Affine $W_a \curvearrowright \mathbb{Z}\Phi$ root lattice

$$W_a = W_f \ltimes \mathbb{Z}\Phi$$ ↳ by translations

Extended (affine) $W_{ext} \curvearrowright X$ weight lattice

$$W_{ext} = W_f \ltimes X$$ ↳ by translations



as affine Kazhdan - Luzstig polynomials

Spherical Hecke algebra

$W_f \rightsquigarrow \mathcal{H}_f$

$W_a \rightsquigarrow \mathcal{H}_a$

$W_{ext} \rightsquigarrow \mathcal{H}_{ext}$

as affine Kazhdan-Lusztig polynomials

Spherical Hecke algebra

$W_f \rightsquigarrow \mathcal{H}_f$

$W_a \rightsquigarrow \mathcal{H}_a$

$W_{ext} \rightsquigarrow \mathcal{H}_{ext}$

$$\tilde{\mathcal{H}} = \text{"}\mathcal{H}_f \backslash \mathcal{H}_{ext} / \mathcal{H}_f\text{"}$$

$$= \{ f \in \mathcal{H}_{ext} \mid h_w f = f = f h_w \ \forall w \in W_f \}$$

as affine Kazhdan – Lusztig polynomials

Spherical Hecke algebra

$W_f \rightsquigarrow \mathcal{H}_f$

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$\leadsto$ has standard basis $\{h_\lambda\}_{X^+}$

& KL – basis $\{\underline{h}_\lambda\}_{X^+}$

$\boxed{\text{Rmk: } W_f \backslash W_e / W_f = X^+}$

as affine Kazhdan – Lusztig polynomials

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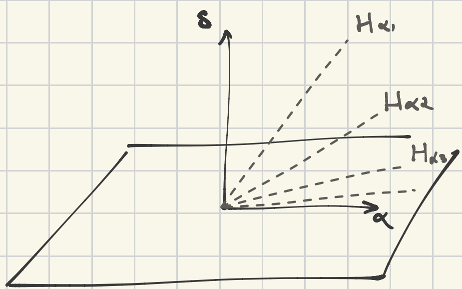
& KL-basis $\{\underline{h}_\lambda\}_{X^+}$

$\boxed{\text{Rmk: } W_f \backslash W_e / W_f = X^+}$

Thm: [Lusztig 83']

$$\underline{h}_\lambda = \sum_{\mu \leq \lambda} K_{\lambda,\mu}(q^2)\, h_\mu$$

In this context, they admit generalization for $\eta \in \hat{X} = X \oplus \mathbb{C}\delta$



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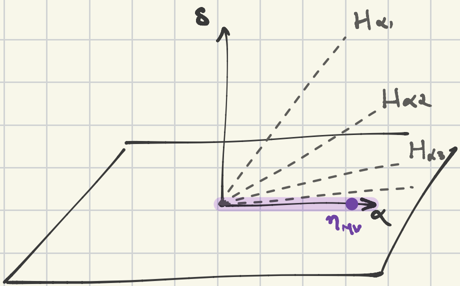
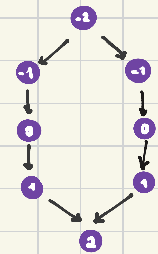


eg. "change" for η^{MV}



$\mathfrak{sl}_3 \curvearrowright Ad = V(\epsilon_1 - \epsilon_3)$

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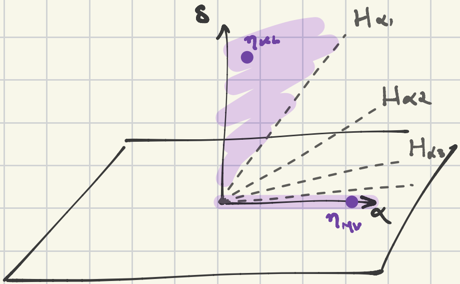
for $\eta \in \hat{X} = X \oplus \mathbb{C}\delta$



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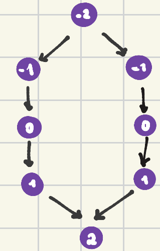

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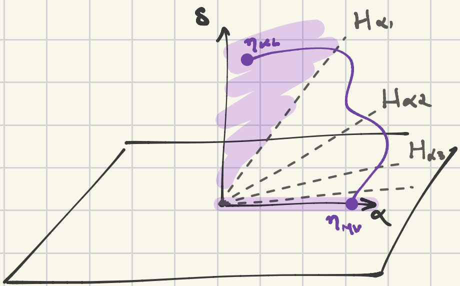
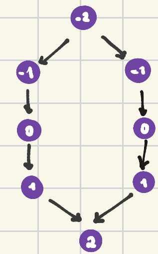


e.g. "change" for η^{MV}



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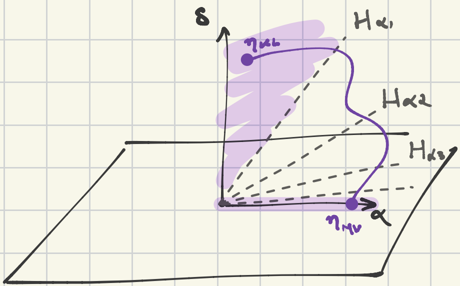

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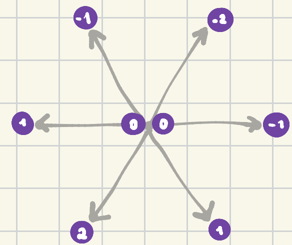

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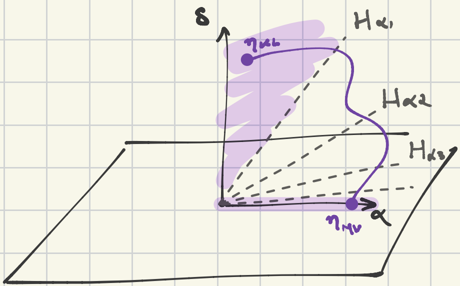

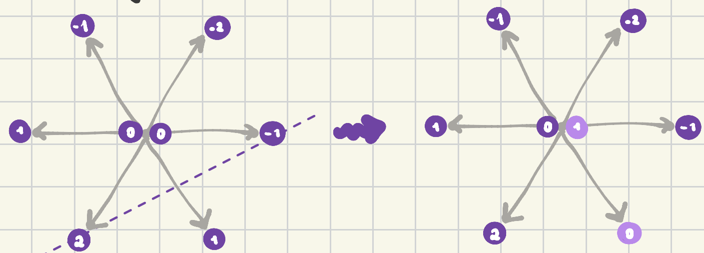

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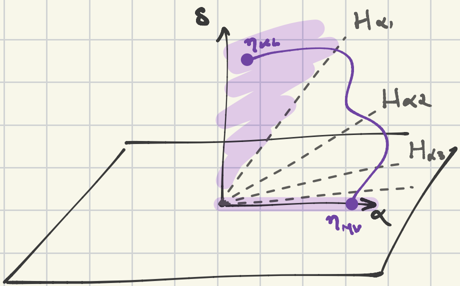

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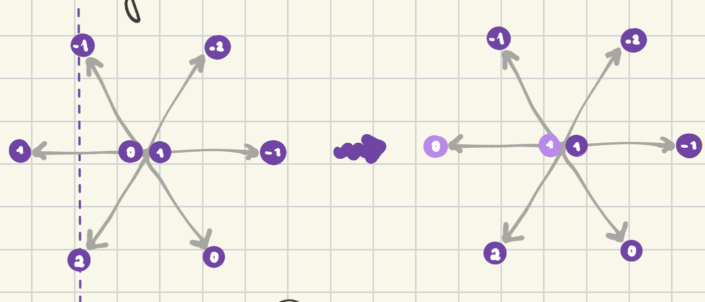

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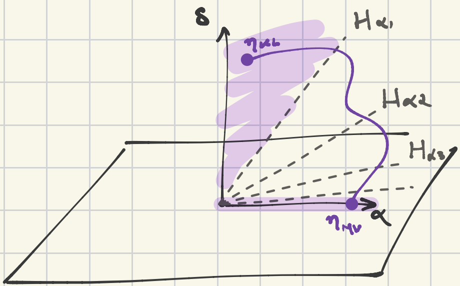

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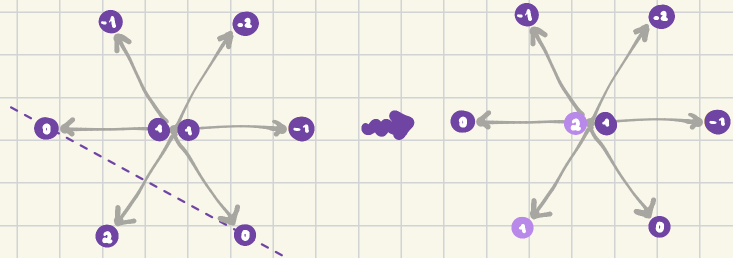

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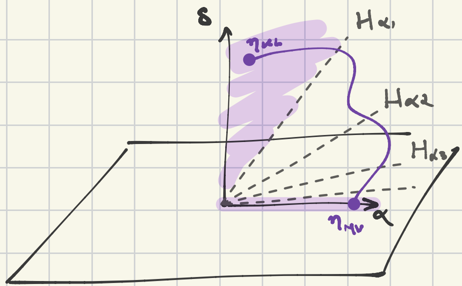

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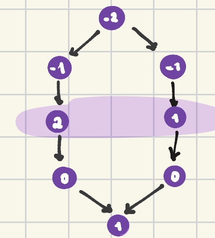

$K_{\lambda_0} = q^2 + q$

$\mathfrak{sl}_3 \curvearrowright Ad = V(\epsilon_1 - \epsilon_3)$

Upshot: To find a charge, it is enough to define swapping functions $\psi_{\alpha^\vee} : \mathcal{B}(\lambda)_\mu \hookrightarrow \mathcal{B}(\lambda)_\nu$ for $\mu < s_{\alpha^\vee}(\mu) = \nu$ satisfying certain conditions

$\leadsto$ We want atomic decompositions!

How to find atomic decompositions?

How to find atomic decompositions?

We need wt preserving embeddings

$$\psi_\alpha : \mathcal{B}(\lambda-\alpha) \hookrightarrow \mathcal{B}(\lambda) \qquad \forall \text{ non-simple roots } \alpha \text{ and } \lambda \text{ big enough}$$

Made with Goodnotes

How to find atomic decompositions?

We need wt preserving embeddings

$$\begin{tikzcd}
 & \mathcal{B}(\lambda-2\alpha) \arrow[dr] & & \\
 & & \mathcal{B}(\lambda-\alpha) \arrow[dr] & \\
\cdots & \mathcal{B}(\lambda-\alpha-\beta) \arrow[ur] \arrow[dr] & & \mathcal{B}(\lambda) \\
 & & \mathcal{B}(\lambda-\beta) \arrow[ur] & \\
 & \mathcal{B}(\lambda-2\beta) \arrow[ur] & &
\end{tikzcd}$$

How to find atomic decompositions?

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 & & \mathcal{B}(\lambda-\beta) \arrow[ur] & \\
 & \mathcal{B}(\lambda-2\beta) \arrow[ur] & &
\end{tikzcd}
$$

ie

$$\operatorname{Im} \psi_\alpha \psi_\beta = \operatorname{Im} \psi_\alpha \cap \operatorname{Im} \psi_\beta = \operatorname{Im} \psi_\beta \psi_\alpha$$

How to find atomic decompositions?

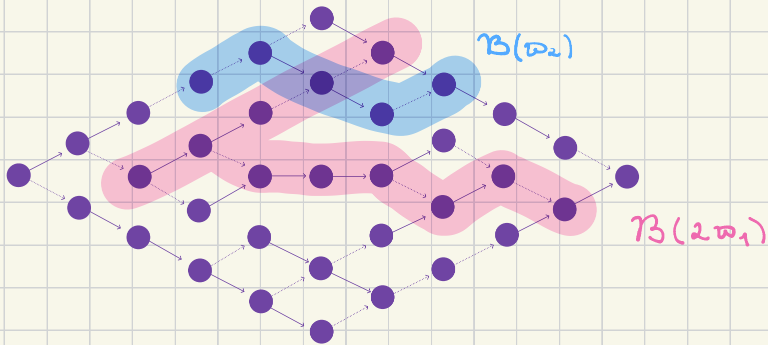
If

$$
\begin{tikzcd}
 & \mathcal{B}(\lambda-\alpha) \arrow[dr] & \\
\mathcal{B}(\lambda-\alpha-\beta) \arrow[ur] \arrow[dr] & \square & \mathcal{B}(\lambda) \\
 & \mathcal{B}(\lambda-\beta) \arrow[ur] &
\end{tikzcd}
\qquad \forall\, \alpha, \beta \in \Phi \setminus \Delta
$$

then $\mathcal{A}(\lambda) = \left(\bigcup_{\alpha} \operatorname{Im} \psi_\alpha \right)^c$

How to find atomic decompositions?

eg $\quad sp_4 \curvearrowright V(2\varpi_1 + \varpi_2)$



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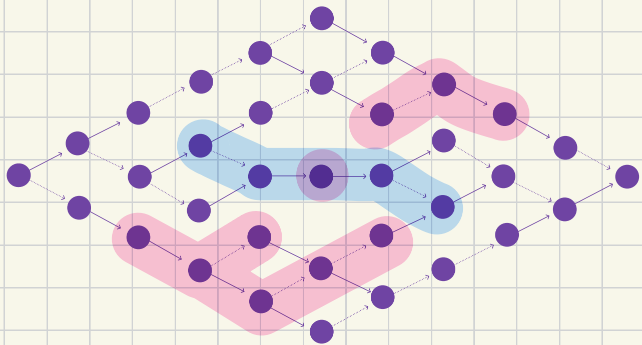

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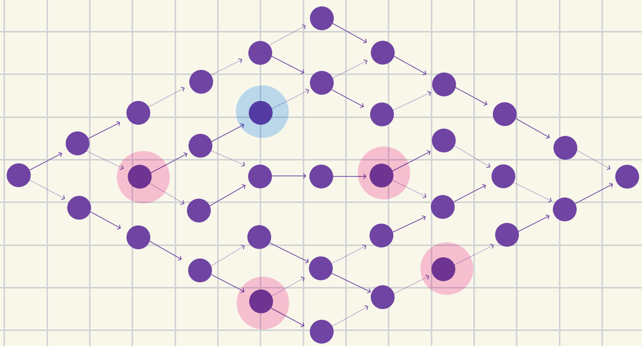

Open question

Is there a theoretical/systematic way of finding wt preserving embeddings

$$
\begin{tikzcd}
 & \mathcal{B}(\lambda-\alpha) \arrow[dr] & \\
\mathcal{B}(\lambda-\alpha-\beta) \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\square"] & & \mathcal{B}(\lambda) \\
 & \mathcal{B}(\lambda-\beta) \arrow[ur] &
\end{tikzcd}
$$

$\forall\, \alpha, \beta \in \Phi \setminus \Delta$

Thank you!