Categorifying orthosympletic Kazhdan–Lusztig polynomials

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University of York

30 Oct 2025

A parable

An alternative parable

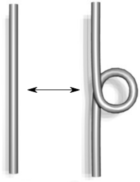
Silly shephards

$$\begin{tikzcd}[column sep=huge]
\mathbb{N} \arrow[r, bend left, "\text{categorification}"] & \mathrm{FinSet} \arrow[l, bend left, "\text{de-categorification}"]
\end{tikzcd}$$

- $| V \amalg W | = | V | + | W |$
  $| V \times W | = | V | \cdot | W |$

Jones polynomial

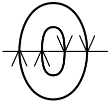
- In the 1980's Jones created a polynomial that was invariant under Reidemeister moves.

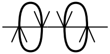


- It is not a complete invariant and it cannot detect the unknot.
- Khovanov 'categorified' the Jones polynomial by creating a cochain of graded vector spaces, H_n^n.
- The Jones polynomial appears in the cohomology of this cochain.
- This can detect the unknot.

Generalised Khovanov arc algebras

- Brudan-Stroppel generalised this algebra, H_n^m, and associated it and it's quasi-hereditary cover to blocks of parabolic cateogry $\mathcal{O}(m,n)$.
- Related to the category of perverse sheaves on the Grassmannian of m-dimensional subspaces in an $(m+n)$-dimensional complex vector space.





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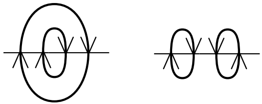

- Ehrig-Stroppel generalised this generalised algebra into type D as well.

Diagrammatic Hecke category

Kahzdan-Lustzig polynomials

- In 1979, Kazhdan and Lusztig introduced the Kazhdan-Lusztig basis of the Hecke algebra of a Coxeter system. The base change matrix between the standard and the Kazhdan-Lusztig basis to be given by a set of polynomials $n_{\mu,\lambda}(q)$.

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- There are many places these polynmials appear all over Representation Theory.

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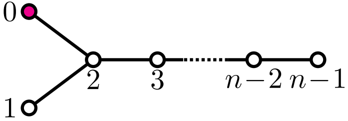
- This proved the famous Kazhdan–Lusztig positivity conjecture
- Williamson also used this structure to find an explicit counter-example to Lustzig's conjecture over $\mathbb{F}_p$.

Maximal parabolic of type (D_n, A_{n-1})

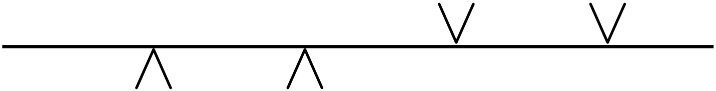
- Take the Weyl group A_{n-1}, the subgroup of D_n ($\leqslant C_n$) generated by the reflections $\{s_1, s_2, ..., s_{n-1}\}$.



- We can visualise the cosets ${}^P W$ of this subgroup via their action permuting n labels from $\{\wedge, \vee\}$.
- In particular s_i acts by permuting the labels in position i and $i+1$ for $1 \leqslant i \leqslant n-1$.
- s_0 permutes and flips (through the horizontal line) the labels in the first and second positions

Example

- For example for n=4, s_2 acts as



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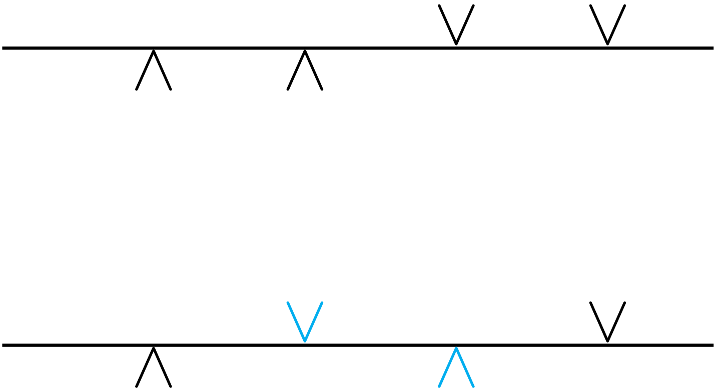

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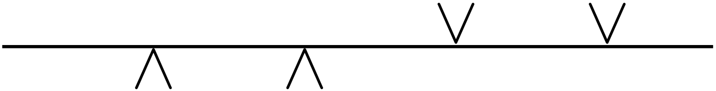

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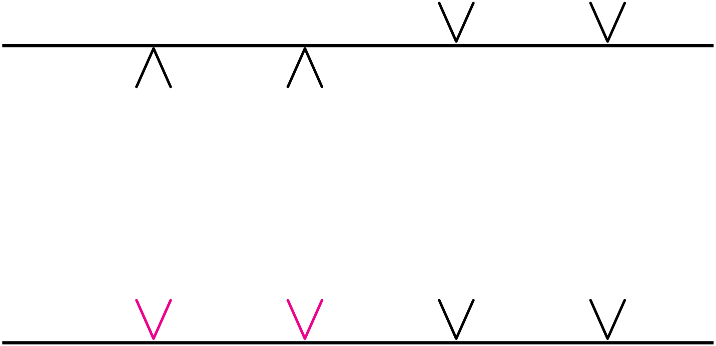
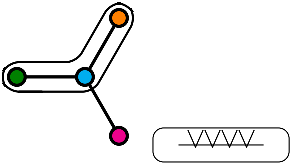


Labelling of cosets



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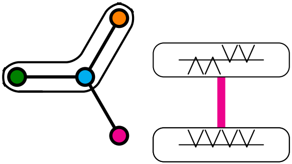

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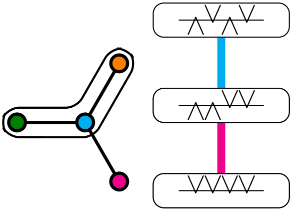

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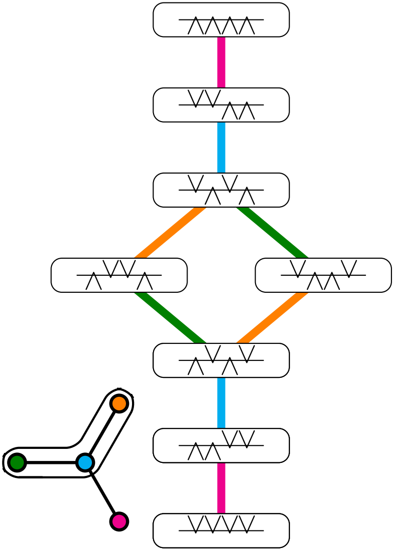
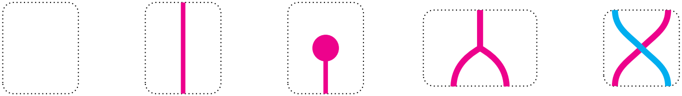


The diagrammatic Hecke category

Definition

Define the **Soergel generators** to be the framed graphs



for reflections $\boldsymbol{\sigma}, \boldsymbol{\tau} \in S_w$ for $m(\boldsymbol{\sigma}, \boldsymbol{\tau}) = 2$. Diagrams made by concatenation are called Soergel diagrams.

The diagrams represent morphisms between Soergel bimodules.

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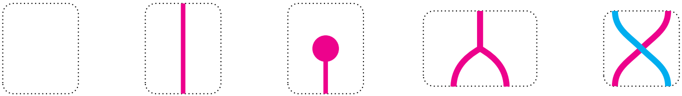

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Definition

The Hecke category, $\mathcal{H}_{(W,P)}$, is spanned by all Soergel diagrams generated by the graphs above modulo a (medium) set of relations. The relations come from the Coxeter-Dynkin diagram of (W, P).

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- Labels of the highest weight modules:

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- Decomposition matrix:

$$n_{\mu,\lambda}(q) = \sum_{k \in \mathbb{Z}} [\Delta(\mu) : L(\lambda)\langle k \rangle] q^k$$

is uni-triangular wrt the partial ordering of $\mathcal{G}_{(W,P)}$.

Calculating Type D Kazhdan–Lusztig polynomials

Cup diagrams

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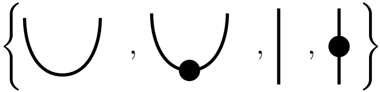
- To a weight $\mu \in {}^{P}W$ we create a cup diagram by drawing cups and strands underneath μ.



Oriented cup diagrams

- A cup diagram is oriented if:
  - For all undecorated cups one label points into the strand and one points out.
  - For all decorated cup connect nodes either both labels point in or both labels point out of the strand.
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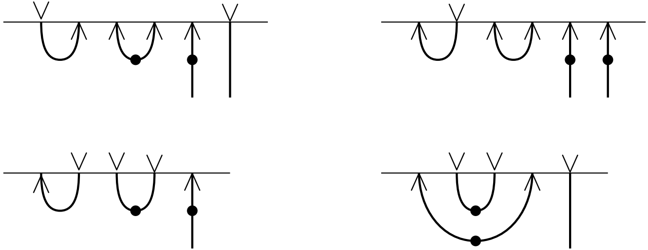

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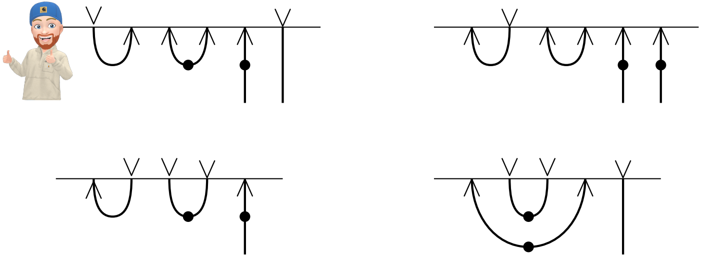

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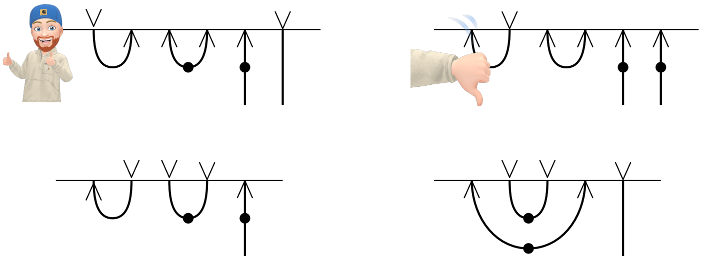

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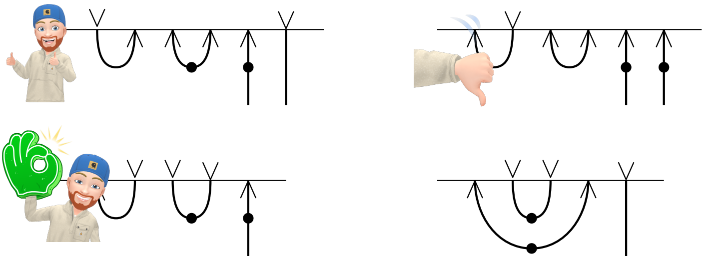

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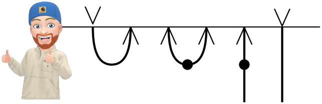

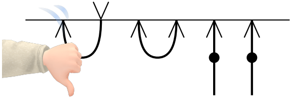

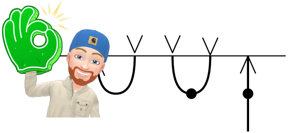

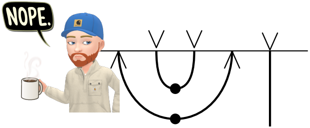

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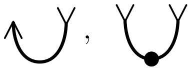
$\sharp \Big\{$  $\Big\}$

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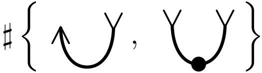

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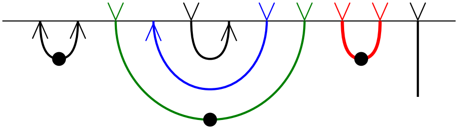
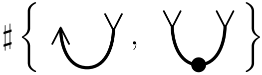


Neutral elements

- For $\mu \in {}^{P}W$ we want to construct the unique maximal cup $\underline{\mu}$ of degree zero.



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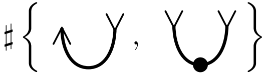

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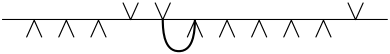

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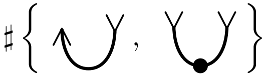

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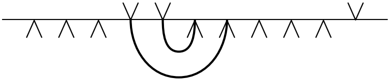

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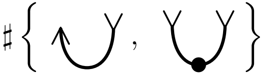

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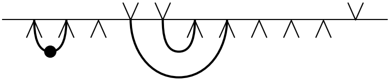

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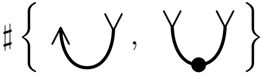

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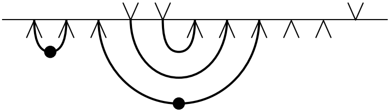

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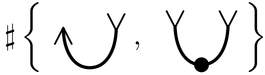

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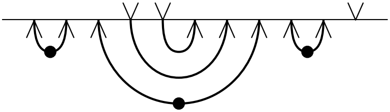

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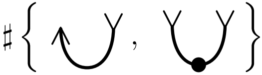

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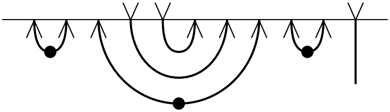

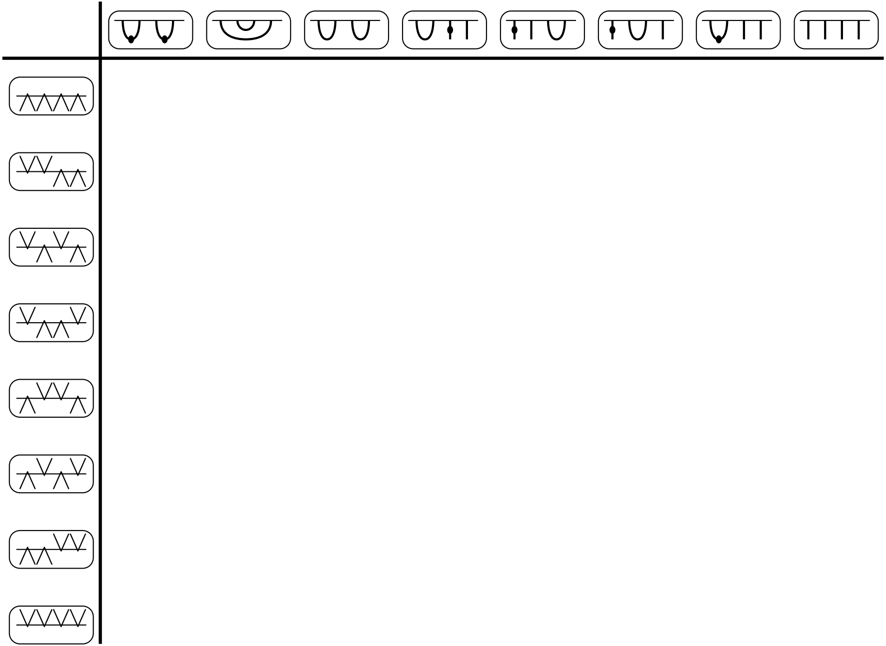

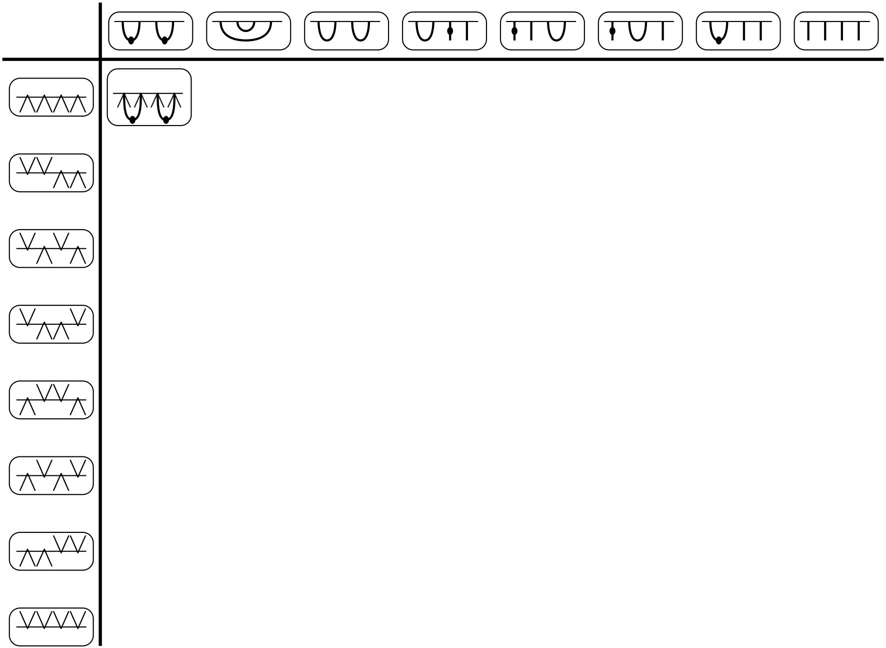

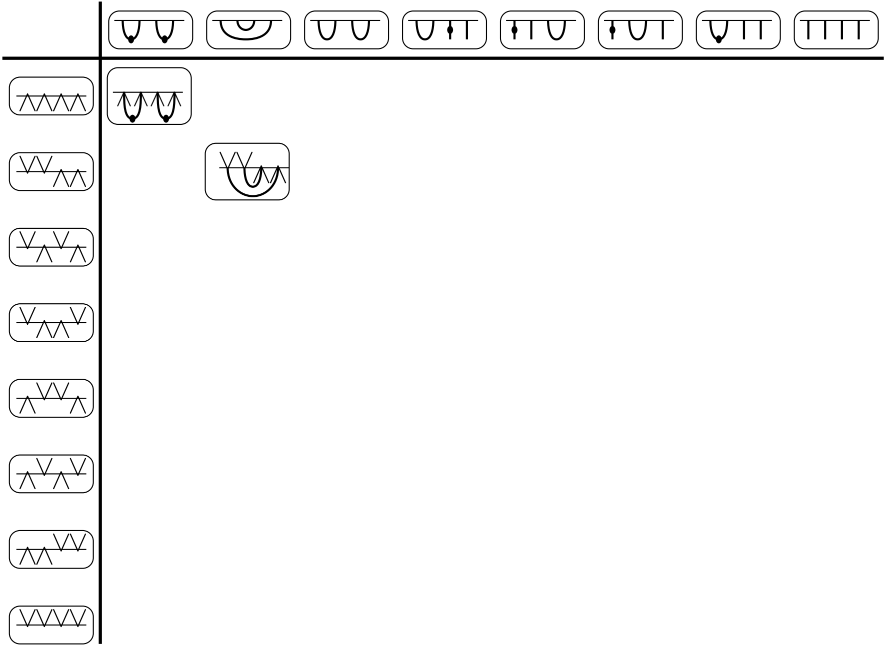

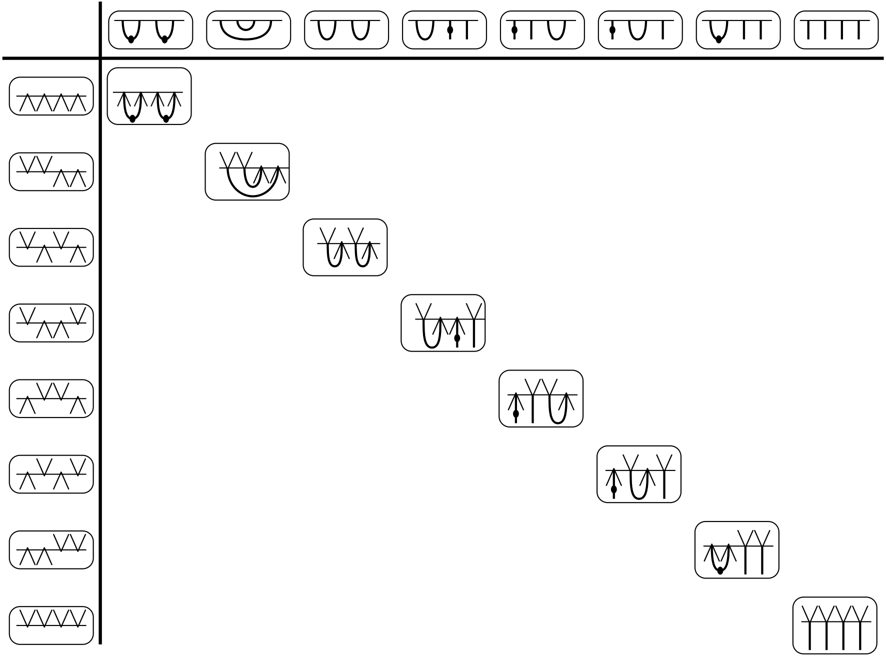

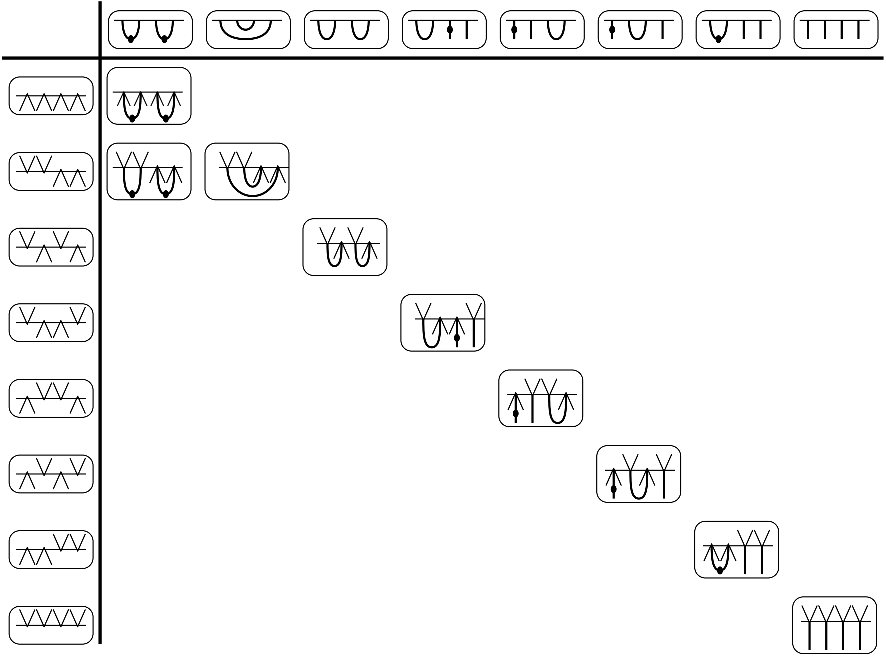

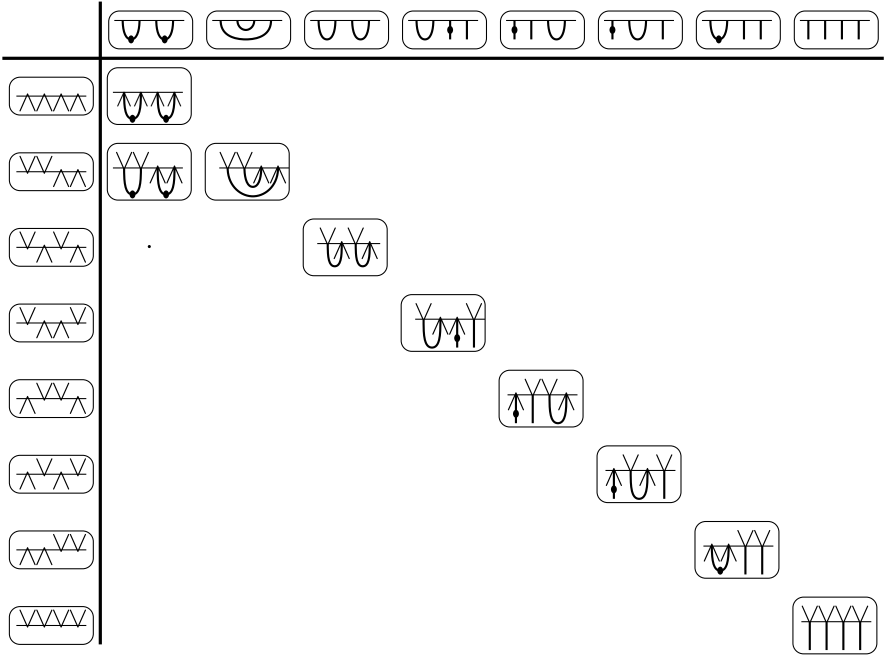

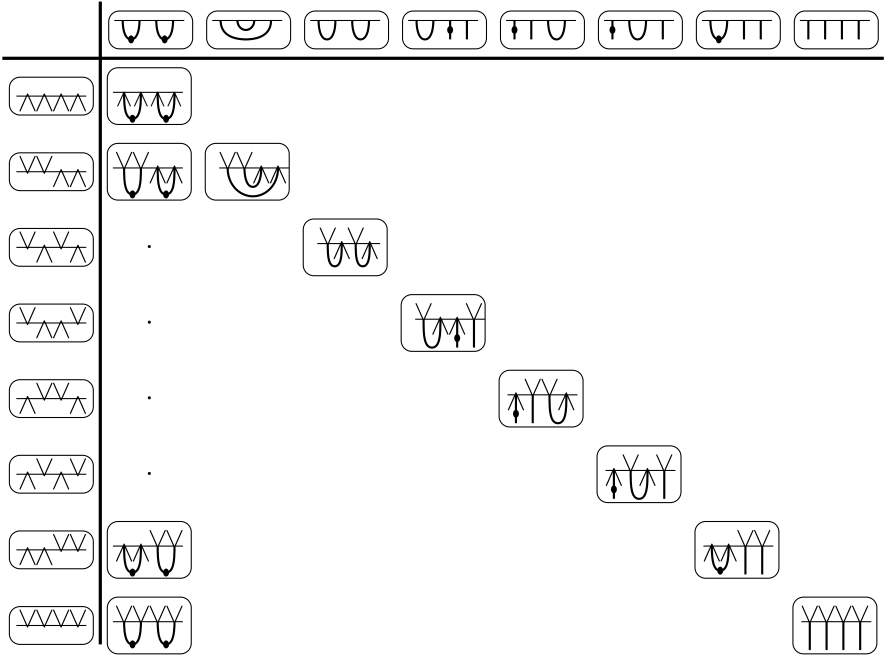

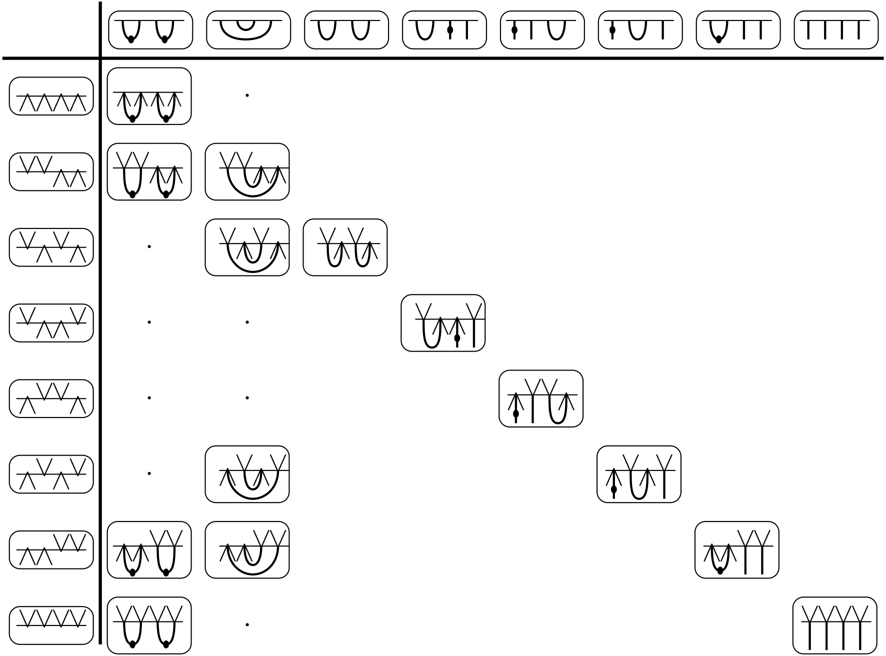

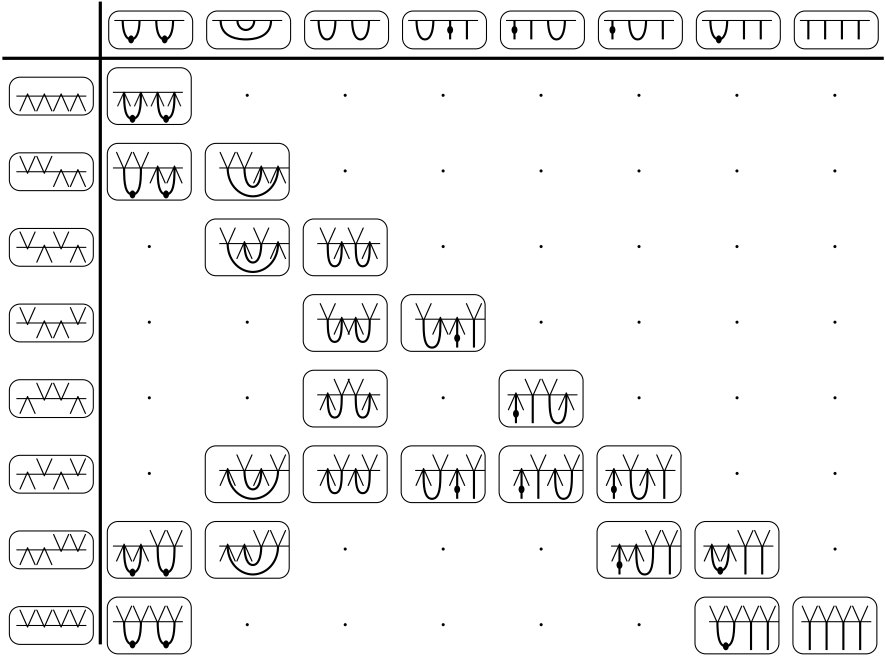

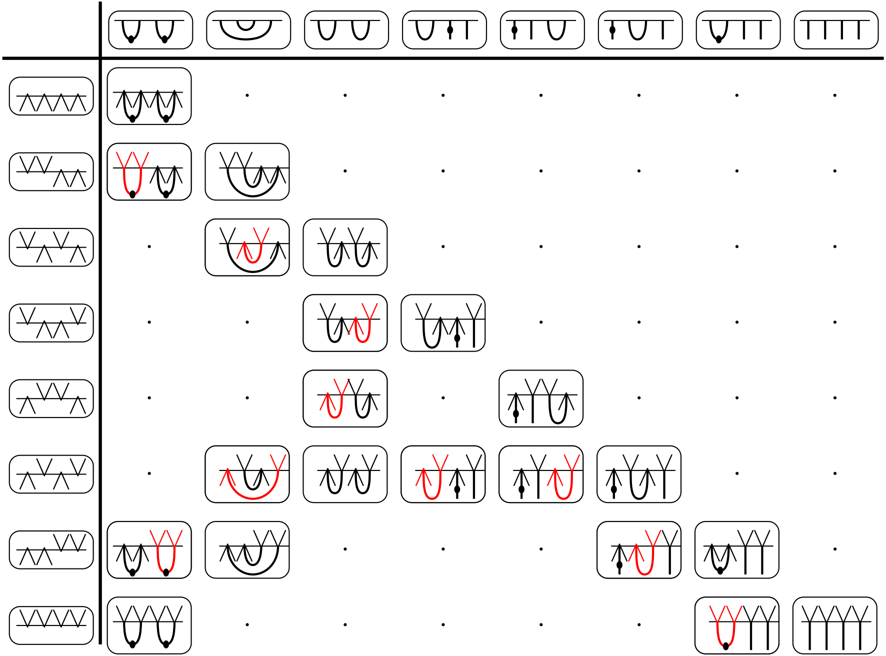

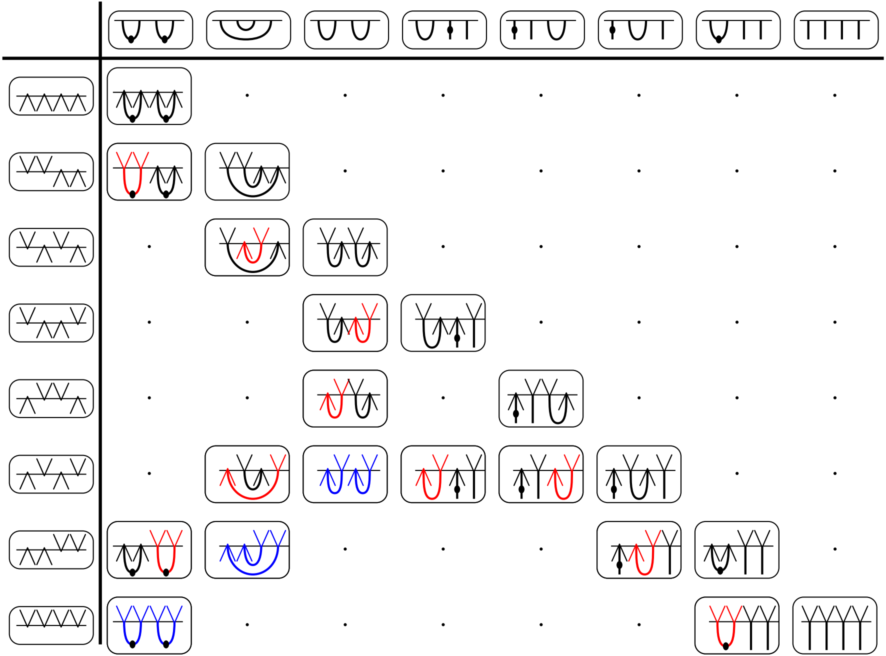

	1	·	·	·	·	·	·	·
	q	1	·	·	·	·	·	·
	·	q	1	·	·	·	·	·
	·	·	q	1	·	·	·	·
	·	·	q	·	1	·	·	·
	·	q	q^2	q	q	1	·	·
	q	q^2	·	·	·	q	1	·
	q^2	·	·	·	·	·	q	1

Flip it

- Every oriented cup diagram of degree k is created by 'flipping' k cups in the diagram $\underline{\mu}\mu$

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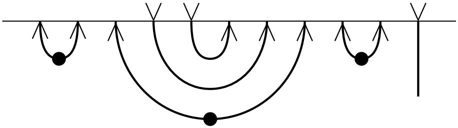

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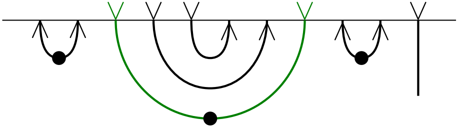

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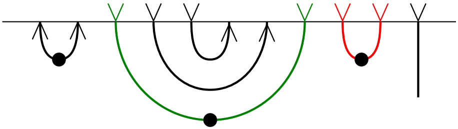

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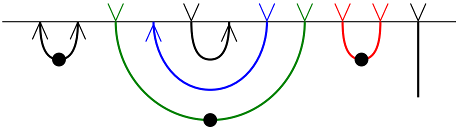

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- It would be 'nice' if this cup flipping could control even more....

Generators

Definition

We say $\lambda = \mu - p$ when $\underline{\lambda}\lambda$ and $\underline{\mu}\mu$ differ by the flipping of a single cup p.

Theorem (M. 25+)

The algebra $\mathcal{H}_{(D_n, A_{n-1})}$ is the associative $\Bbbk$-algebra generated by the elements

$$\{\underline{\mu}\lambda \mid \lambda, \mu \in {}^{P}W \ \textit{with}\ \lambda = \mu - p \ \textit{for some}\ p \in \underline{\mu}\} \cup \{\underline{\mu}\mu \mid \mu \in {}^{P}W\}$$

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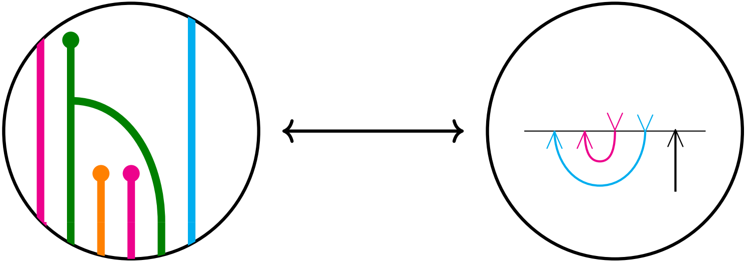
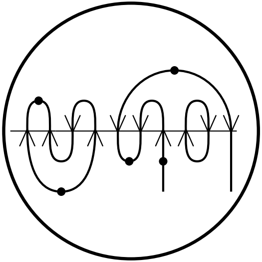
- This presentation allows us to define an explicit isomorphism into the generalised Khovanov arc algebra of type D, $\mathbb{D}_n$.

  We hope to utilise this to provide a faithful quasi-hereditary cover for $\mathbb{D}_n$.

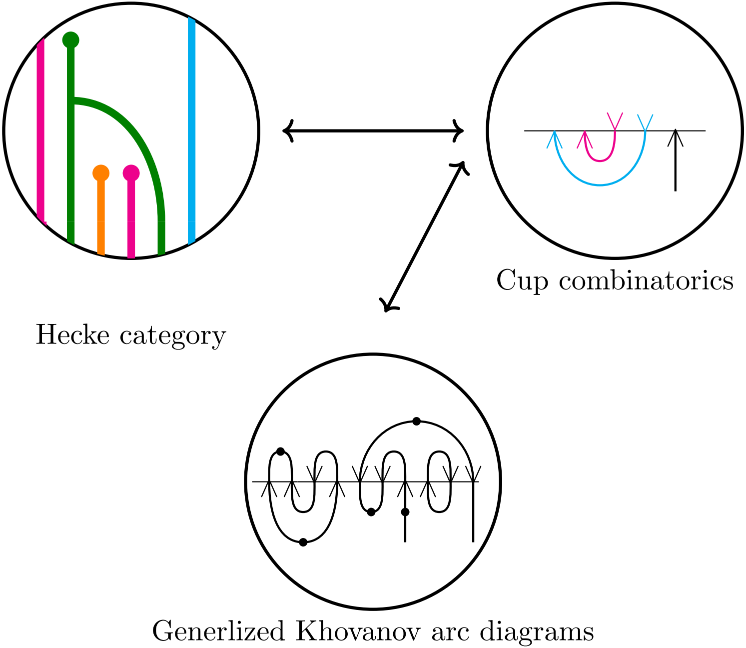


Cup combinatorics

Hecke category



Generlized Khovanov arc diagrams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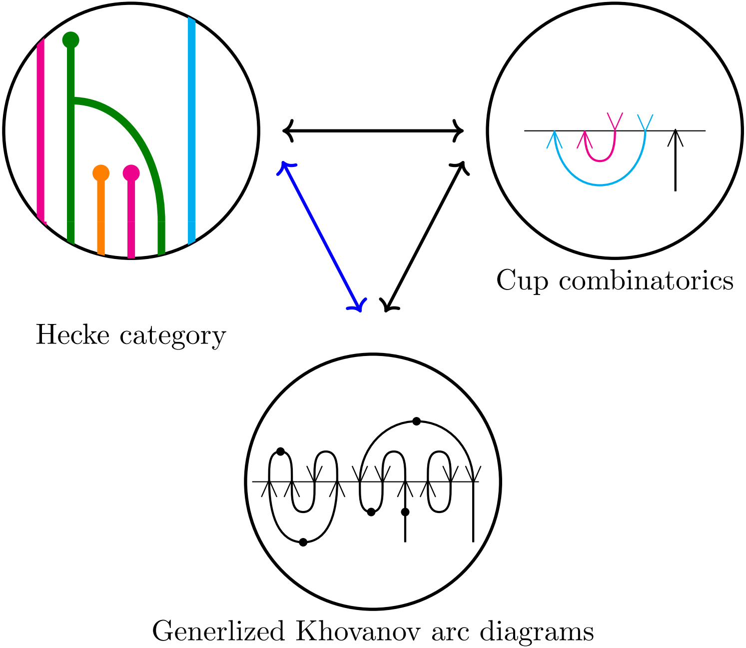
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  - One dimensional simple socle $L(\mu_n)$, indexed by a unique element of maximal degree μ_n.
  - One dimension simple head $L(\mu)$.
  - An extension only when you can 'flip' one cup to get between two cups diagrams:
    We have edges
    $$L_n(\mu) \longrightarrow L_n(\nu)$$
    in an Alperin diagram whenever $\mu = \nu \pm p$ for some $p \in \underline{\nu}$ or $p \in \underline{\mu}$.
- Reminder: the composition factors $[\Delta(\mu) : L(\mu)]\langle k \rangle$ give us the Kazdhan-Lusztig polynomials.

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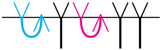

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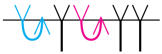

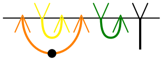

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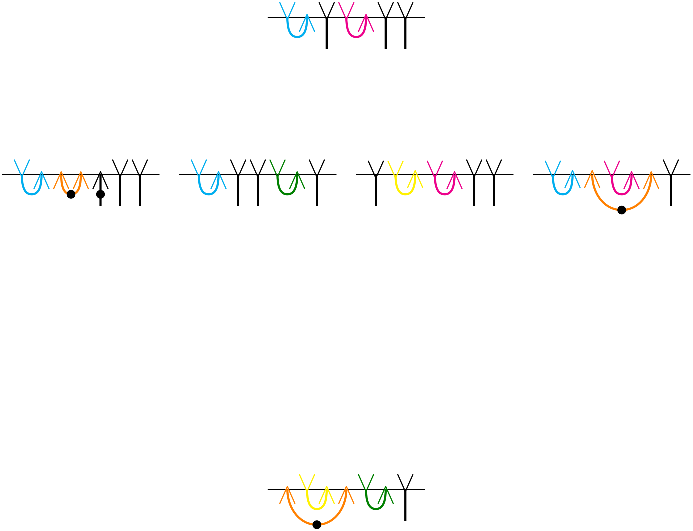

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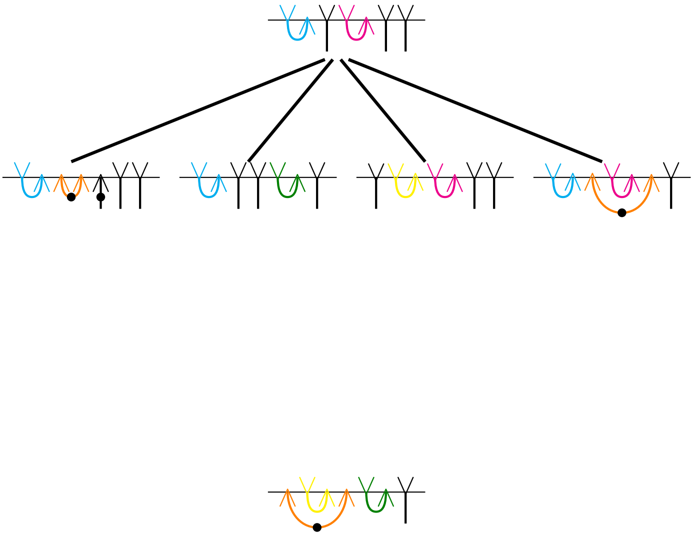

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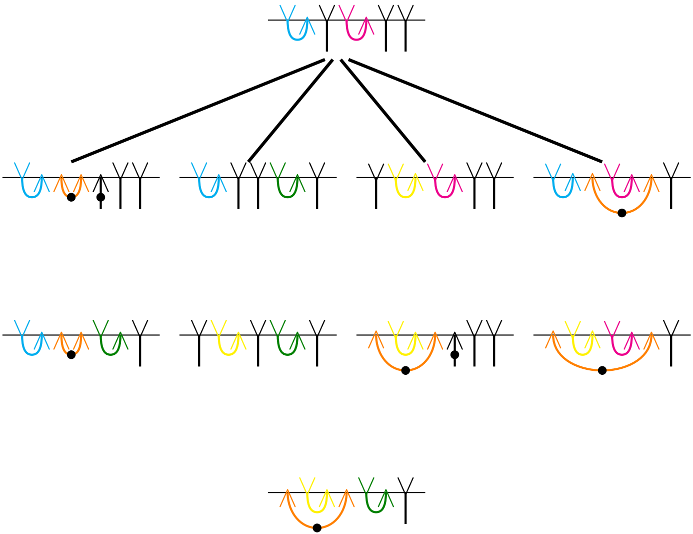

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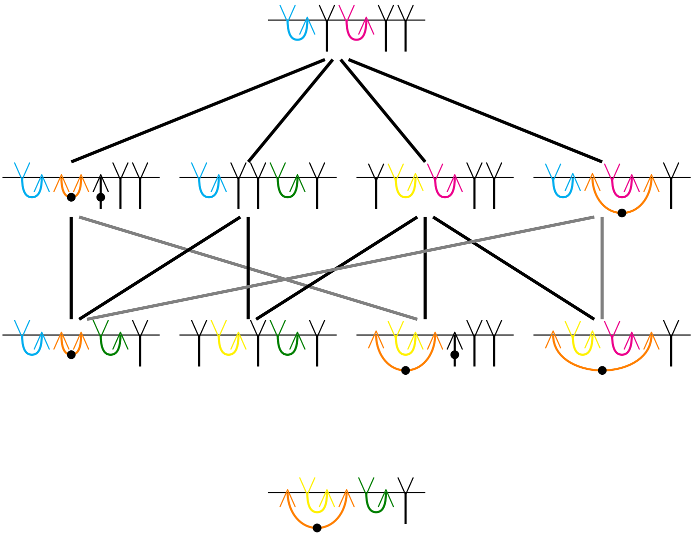

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