From Cayley-Hamilton to Trace Identities: New Insights into Upper Triangular Matrices

(based on a joint work with Antonio Ioppolo)

Collaborations in Algebra, Representation theory and Ethics

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Polynomial Identities

Notation and conventions:

- F is a field of characteristic zero;
- A is an associative algebra;
- $X := \{x_1, x_2, \dots\}$ is a countable set;
- $F\langle X\rangle$ is the free associative algebra over F generated by X.

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Definition

An element $f(x_1, \dots, x_n)$ of $F\langle X\rangle$ is a **polynomial identity** (denoted as $f \equiv 0$) for A if $f(a_1, \dots, a_n) = 0_A$ for every $a_1, \dots, a_n \in A$.
A is a **PI-algebra** if A satisfies a non-trivial polynomial identity $f \neq 0_{F\langle X\rangle}$.
Set $\mathrm{Id}(A) := \{f \mid f \in F\langle X\rangle,\ f \ PI \text{ for } A\}$.

Examples of Identities

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$$[x_1, x_2] := x_1x_2 - x_2x_1 \equiv 0;$$

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$$[[x_1, x_2]^2, x_3] \equiv 0 \quad \text{(Hall's identity)};$$

- the algebra of $n \times n$ matrices, $M_n(F)$:

$$St_{2n}(x_1, \dots, x_{2n}) := \sum_{\sigma \in S_{2n}} \text{sgn}(\sigma) x_{\sigma(1)} \cdots x_{\sigma(2n)} \equiv 0 \quad \text{(Amitsur–Levitzki theorem)}.$$

Infinite dimensional (non-commutative) PI-algebra

Let E be the infinite-dimensional Grassmann algebra
$E := \langle 1_F, \epsilon_1, \epsilon_2, \dots | \epsilon_i \epsilon_j = -\epsilon_j \epsilon_i, \ \forall i, j \geq 1 \rangle$.

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- The algebra E has the following decomposition

$$E = E^{(0)} \oplus E^{(1)},$$

$$E^{(0)} = \mathrm{span}_F \{ \epsilon_{i_1} \dots \epsilon_{i_{2l}} | \ l \in \mathbb{N}, \ 1 \leq i_1 < \dots < i_{2l} \},$$
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- $E^{(0)} = Z(E)$;
- $[[x_1, x_2], x_3]$ is a polynomial identity for E.

Historical Role of PI-theory I

Kurosh Problem

Is every algebraic algebra locally finite? ✗

Bounded Kurosh Problem

Is every algebra in which each element is the root of some non-trivial polynomial of some fixed degree n locally finite? ✓

Can we find a stronger result?

Note: If A satisfies the *hypotheses* of the Bounded Kurosh Problem, then A satisfies a *polynomial identity*.

PI Kurosh Problem

Is every algebraic algebra which satisfies a polynomial identity locally finite? ✓

Historical Role of PI-theory II

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Find necessary and sufficient conditions for which a ring R can be embedded into a matrix algebra $M_n(B)$, over some commutative ring B.

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Historical Role of PI-theory II

1 Introduction

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Find necessary and sufficient conditions for which a ring R can be embedded into a matrix algebra $M_n(B)$, over some commutative ring B.

Example: $F\langle x_1, \ldots, x_n \rangle$ cannot be embedded in any matrix algebra (Malcev).
In fact, it does not satisfy any polynomial identity, while matrix algebras are PI-algebras.
Fact: There exist algebras satisfying all the polynomial identities of matrices that cannot be embedded in a matrix algebra.

Finiteness Problem

$$\mathrm{Id}(M_2(F)) \supseteq \{[[x_1, x_2]^2, x_3], St_4(x_1, x_2, x_3, x_4)\}.$$

Finiteness Problem
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$$\mathrm{Id}(M_2(F)) = \langle [[x_1, x_2]^2, x_3], St_4(x_1, x_2, x_3, x_4) \rangle_T.$$

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Kemer's Theorem, 1987 (Specht's Problem, 1950)

Char F=0, A is a PI-algebra $\Rightarrow$ $\mathrm{Id}(A)$ is finitely generated as a T-ideal.

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Open Problem

$$\mathrm{Id}(M_n(F)) = \langle\ ?\ \rangle_T, n > 2$$

Let's change perspective

2 Trace Identities

Let us come back to the algebra of 2×2 matrices, $M_2(F)$. Let $a \in M_2(F)$:

- $a^2 - tr(a)a + det(a)\mathbb{I}_2 = 0 \quad$ (Cayley-Hamilton Theorem),

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If we add a **formal symbol** *Tr* to the set of polynomials $F\langle X\rangle$:

$$x^2 - Tr(x)x + \frac{1}{2}(Tr(x)^2 - Tr(x^2)) \equiv 0 \text{ in } M_2(F).$$

Definition

An algebra with trace (A, tr) is an F-algebra A endowed with a linear map $\mathrm{tr} : A \to F$ such that, for every $a, b \in A$,

$$\mathrm{tr}(ab) = \mathrm{tr}(ba).$$

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- If t is a trace on the algebra D_n of $n \times n$ diagonal matrices, then there exist scalars $\alpha_1, \dots, \alpha_n \in F$ such that, if $a = \mathrm{diag}(a_{11}, \dots, a_{nn}) \in D_n$, then

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$$\mathrm{t}(a) = \alpha_1 a_{11} + \cdots + \alpha_n a_{nn} =: t_{\alpha_1, \dots, \alpha_n}(a).$$

Trace Polynomials

2 Trace Identities

$F\langle X, \mathrm{Tr}\rangle$ denotes the set of trace polynomials, i.e., it is the algebra generated by $F\langle X\rangle$ and the set of central variables $\mathrm{Tr}(M)$ for every monomial M, subject to the conditions

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Examples of polynomials: $x_1 \,\mathrm{Tr}(x_2x_3) + x_3 \,\mathrm{Tr}(x_2)\,\mathrm{Tr}(x_1)$, $x_1^2x_2x_3 + \mathrm{Tr}(x_1x_2x_3x_2)$.

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Definition

$f(x_1, \ldots, x_n, \mathrm{Tr}) \in F\langle X, \mathrm{Tr}\rangle$ is a trace identity of (A, tr) if, after substituting the x_i's with arbitrary elements $a_i \in A$ and the formal trace Tr with tr, we obtain 0.

$\mathrm{Id}^{tr}(A)$ denotes the set of **trace identities** of A, $\mathrm{Id}^{\mathrm{ptr}}(A)$ denotes the set of **pure trace identities**, namely identities in which all variables appear inside traces.

We have already noticed that $x^2 - Tr(x)x + \frac{1}{2}(Tr(x)^2 - Tr(x^2)) \equiv 0$ on $(M_2(F), t_1)$.

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Cayley-Hamilton Polynomial

For every $n \geq 2$, we can define a polynomial $C_n(x, Tr)$ that can be obtained from the Cayley-Hamilton theorem and such that

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Theorem (C. Procesi 1976, Ju. P. Razmyslov 1974)

$$\mathrm{Id}^{\mathrm{tr}}(M_n(F), t_1) = \langle C_n(x, Tr) \rangle_T.$$

Proportional traces

Let $\alpha \in F \setminus \{0\}$ and consider a *monomial* m with s traces in it. Define

$$\varphi_\alpha : F\langle X, \mathrm{Tr}\rangle \to F\langle X, \mathrm{Tr}\rangle, \quad m \mapsto \alpha^{-s} m.$$

(A. Giambruno, A. Ioppolo, D. La Mattina, 2023)

Consider (A, t) and the corresponding algebra (A, t_α) with proportional trace, namely $t_\alpha = \alpha t$. Then

$$f \in \mathrm{Id}^{\mathrm{tr}}(A, t) \iff \varphi_\alpha(f) \in \mathrm{Id}^{\mathrm{tr}}(A, t_\alpha).$$

Corollary: $\mathrm{Id}^{\mathrm{tr}}(M_n(F), t_\alpha) = \langle \varphi_\alpha(C_n(x, \mathrm{Tr}))\rangle$.

Motivations

Why study "algebras with trace" and "trace identities"?

- To obtain information on (ordinary) identities;
- Connections with "Embedding Problems".

Malcev Problem

Find necessary and sufficient conditions for which a ring R can be embedded into a matrix algebra $M_n(B)$, over some commutative ring B.

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Theorem [C. Procesi]

An **algebra with trace** A satisfies a formal Cayley–Hamilton polynomial of degree n if, and only if, it can be embedded in $n \times n$ matrices over a commutative algebra.

Some Tools

Definition

- $\mathrm{Id}^{tr}(A)$ is completely determined by $\{MT_n \cap \mathrm{Id}^{\mathrm{Tr}}(A)\}_{n\geq 1}$, where MT_n, denotes the vector space of multilinear trace polynomial of degree n. E.g. $x_1 \operatorname{Tr}(x_2x_3),\ x_3 \operatorname{Tr}(x_2)\operatorname{Tr}(x_1) \in MT_3$
- $c_n^{\mathrm{tr}}(A) = \dim_F \frac{MT_n}{MT_n \cap \mathrm{Id}^{\mathrm{tr}}(A)}$ is called n-th trace codimension of A.

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Theorem [A. Giambruno, A. Ioppolo, D. La Mattina, 2022]

Let $A = A_{ss} + J$ be a finite-dimensional unitary algebra with trace tr and let $\mathrm{tr}(J) = 0$. The trace exponent of A

$$\exp^{\mathrm{tr}}(A) = \lim_{n\to\infty} \sqrt[n]{c_n^{\mathrm{tr}}(A)}$$

exists and it is an integer.

State of Art about Trace Identities

- $\mathrm{Id}^{\mathrm{tr}}(M_n(F))$ fully described (C. Procesi, Ju. P. Razmyslov) ✓
- $\mathrm{Id}^{\mathrm{tr}}(D_2(F))$ fully described and $\mathrm{Id}^{tr}(D_n(F))$ with partial results (A. Berele, A. Giambruno, A. Ioppolo, P. Koshlukov, D. La Mattina) ✓
- Let UT_n be the algebra of $n \times n$ upper triangular matrices over the field F. $\mathrm{Id}^{\mathrm{tr}}(UT_n(F))$ fully described with **usual trace** (A. Berele) ✓

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- Let UT_n be the algebra of $n \times n$ upper triangular matrices over the field F. $\mathrm{Id}^{\mathrm{tr}}(UT_n(F))$ fully described with **usual trace** (A. Berele) ✓
- $\mathrm{Id}^{\mathrm{tr}}(UT_n(F))$ with other traces...

Some Identities on $UT_n(F)$

Theorem [Y. N. Mal'tsev, 1971]

$$\mathrm{Id}(UT_n(F)) = \langle [x_1, x_2][x_3, x_4] \cdots [x_{2n-1}, x_{2n}] \rangle_T .$$

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Theorem [A. Berele, 1996]

Let t_1 be the usual trace, then $\mathrm{Id}^{\mathrm{Tr}}((UT_n(F), t_1)$ is generated by

1. $\mathrm{Tr}(x_1[x_2, x_3])$,
2. $[x_1, x_2][x_3, x_4] \cdots [x_{2n-1}, x_{2n}]$,
3. $C_n(x, \mathrm{Tr})$,
4. $n - 1$ trace polynomials that can be thought as interpolating between 2. and 3.

Traces on $UT_n(F)$

Remark

$UT_n = D_n + J$, where $J = \mathrm{span}_F\{e_{ij} \mid 1 \leq i < j \leq n\}$. In particular, if tr is a trace on the algebra UT_n, then tr *vanishes* on J.

Proposition [A. Ioppolo, E.P]

Let tr be a trace on $UT_n(F)$. Then there exist $\alpha_1, \ldots, \alpha_n \in F$ such that $\mathrm{tr} = t_{\alpha_1,\ldots,\alpha_n}$.

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Proposition [A. Ioppolo, E.P]

$$\text{Id}^{\text{ptr}}(D_n, t_{\alpha_1,\ldots,\alpha_n}) = \text{Id}^{\text{ptr}}(UT_n, t_{\alpha_1,\ldots,\alpha_n}).$$

UT_n with Weak-degenerate Traces

Theorem [A. Ioppolo, E.P]

Assume that $\alpha_i = 0$, for some $i \in \{1, \dots, n\}$. Then, every trace identity of $(UT_n(F), t_{\alpha_1,\dots,\alpha_n})$, that is not a consequence of the commutator $[x_1, x_2]$, is a *consequence* of pure trace identities.

Theorems [A. Ioppolo, E.P]

Let $\alpha, \beta \neq 0$, $\alpha \neq \beta$,

1. $c_n^{tr}(UT_2, t_{\alpha,\alpha}) = 2^n + 2^{n-1}(n-2) + 1$.
2. $\mathrm{Id}^{Tr}(UT_2, t_{\alpha,0}) = \langle [x_1, x_2][x_3, x_4], \mathrm{Tr}(x_1)\mathrm{Tr}(x_2) - \alpha \mathrm{Tr}(x_1 x_2), (\mathrm{Tr}(x_1) - \alpha x_1)[x_2, x_3] \rangle_{T^{Tr}}$

Furthermore,

$$c_n^{tr}(UT_2, t_{\alpha,0}) = 2^n + 2^{n-1}(n-2) + 1.$$

3. $\mathrm{Id}^{Tr}(UT_2, t_{\alpha,\beta})$ is generated by:

$$[x_1, x_2][x_3, x_4], \quad \mathrm{Tr}(x_1)[x_2, x_3] - \alpha x_1[x_2, x_3] - \beta [x_2, x_3]x_1,$$
$$\mathrm{Tr}([x_1, x_2]x_3), \quad \beta f_4 - f_5, \quad f_4 + \alpha\beta[x_2, [x_1, x_3]],$$

where $\mathrm{Id}^{tr}(D_2, t_{\alpha,\beta}) = \langle [x_1, x_2], f_4, f_5 \rangle_{T^{tr}}$. Furthermore,

$$c_n^{tr}(UT_2, t_{\alpha,\beta}) = 2^{n-1}(n-2) + 2^{n+1} - n.$$

Pure Trace Identities for UT_2

Theorems [A. Ioppolo, E.P]

Let $\alpha, \beta \neq 0$, $\alpha \neq \beta$,

1. $\mathrm{Id}^{\mathrm{ptr}}(UT_2, t_{\alpha,\alpha})$ is generated by $\mathrm{Tr}(x_1[x_2, x_3])$, $\mathrm{Tr}(\varphi_\alpha(C_n(x_1, x_2, \mathrm{Tr}))x_3)$ and
$$c_n^{\mathrm{ptr}}(UT_2, t_{\alpha,\alpha}) = 2^{n-1}.$$
2. $\mathrm{Id}^{\mathrm{ptr}}(UT_2, t_{\alpha,0})$ is generated by $\mathrm{Tr}(x_1)\,\mathrm{Tr}(x_2) - \alpha\,\mathrm{Tr}(x_1x_2)$ and
$$c_n^{\mathrm{ptr}}(UT_2, t_{\alpha,0}) = 1.$$
3. $\mathrm{Id}^{\mathrm{ptr}}(UT_2, t_{\alpha,\beta})$ is generated by: $Tr(x_1[x_2, x_3])$, $\mathrm{Tr}(f_4x_4)$, $\mathrm{Tr}(f_5x_4)$, and
$$c_n^{\mathrm{ptr}}(UT_2, t_{\alpha,\beta}) = 2^n - n.$$

Thank you for listening!