Hecke modifications and parabolic bundles

Inder Kaur

University of Glasgow

CARE, Lyon - October 2025

Hecke Modification at a single point

- X smooth, projective curve over a field k.
- Let E be a vector bundle of rank n on X and $p \in X$ a closed point.
- A *Hecke modification of E at p* (of weight r), denoted $\alpha : E' \xrightarrow[p]{r} E$ is a vector bundle E' of rank n such that E and E' fit in the following short exact sequence

$$0 \to E' \to E \to {\mathcal{O}_p}^{\oplus r} \to 0$$

where $\mathcal{O}_p$ is the structure sheaf of the point p.

- We say E is *Hecke modified to E'* at the point p.

Hecke Modification at a single point

- X smooth, projective curve over a field k.
- Let E be a vector bundle of rank n on X and $p \in X$ a closed point.
- A *Hecke modification of E at p* (of weight r), denoted $\alpha : E' \xrightarrow[p]{r} E$ is a vector bundle E' of rank n such that E and E' fit in the following short exact sequence

$$0 \to E' \to E \to {\mathcal{O}_p}^{\oplus r} \to 0$$

where $\mathcal{O}_p$ is the structure sheaf of the point p.

- We say E is *Hecke modified to E'* at the point p.

Hecke Modification at a single point

- X smooth, projective curve over a field k.
- Let E be a vector bundle of rank n on X and $p \in X$ a closed point.
- A *Hecke modification of* E *at* p (of weight r), denoted $\alpha : E' \xrightarrow[p]{r} E$ is a vector bundle E' of rank n such that E and E' fit in the following short exact sequence

$$0 \to E' \to E \to {\mathcal{O}_p}^{\oplus r} \to 0$$

where $\mathcal{O}_p$ is the structure sheaf of the point p.

- We say E is *Hecke modified to* E' at the point p.

Hecke Modification at a single point

- X smooth, projective curve over a field k.
- Let E be a vector bundle of rank n on X and $p \in X$ a closed point.
- A *Hecke modification of E at p* (of weight r), denoted $\alpha : E' \xrightarrow[p]{r} E$ is a vector bundle E' of rank n such that E and E' fit in the following short exact sequence

$$0 \to E' \to E \to {\mathcal{O}_p}^{\oplus r} \to 0$$

where $\mathcal{O}_p$ is the structure sheaf of the point p.

- We say E is *Hecke modified to E'* at the point p.

Properties of a Hecke Modification

- For the Hecke modification

$$0 \to E' \to E \to \mathcal{O}_p{}^{\oplus r} \to 0$$

we have:

- E is of rank n, then E' is of rank n.
- E is of degree d, then E' is of degree $d - r$.
- The weight r is at most n, i.e. $r \leq n$.

Properties of a Hecke Modification

- For the Hecke modification

$$0 \to E' \to E \to \mathcal{O}_p{}^{\oplus r} \to 0$$

we have:

- E is of rank n, then E' is of rank n.
- E is of degree d, then E' is of degree $d - r$.
- The weight r is at most n, i.e. $r \leq n$.

Properties of a Hecke Modification

- For the Hecke modification

$$0 \to E' \to E \to {\mathcal{O}_p}^{\oplus r} \to 0$$

we have:

- E is of rank n, then E' is of rank n.
- E is of degree d, then E' is of degree $d - r$.
- The weight r is at most n, i.e. $r \leq n$.

Properties of a Hecke Modification

- For the Hecke modification

$$0 \to E' \to E \to \mathcal{O}_p{}^{\oplus r} \to 0$$

we have:

- E is of rank n, then E' is of rank n.
- E is of degree d, then E' is of degree $d - r$.
- The weight r is at most n, i.e. $r \leq n$.

Equivalence classes of Hecke modifications

- Two Hecke modifications $\alpha' : E' \xrightarrow[p]{r} E$ and $\alpha'' : E'' \xrightarrow[p]{r} E$ at a point p are *equivalent* if there exists an isomorphism $\phi : E' \to E''$ such that $\alpha' = \alpha'' \circ \phi$ i.e.

$$\begin{tikzcd}
0 \arrow[r] & E' \arrow[r, "\alpha'"] \arrow[d, "\cong\ \phi"'] & E \arrow[r] \arrow[d] & \mathcal{O}_p^{\oplus r} \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & E'' \arrow[r, "\alpha''"] & E \arrow[r] & \mathcal{O}_p^{\oplus r} \arrow[r] & 0
\end{tikzcd}$$

commutes.

- Denote by $\mathcal{H}_r(X, E, p)$ the set of equivalence classes of Hecke modifications of E of weight r at the point p.

Equivalence classes of Hecke modifications

- Two Hecke modifications $\alpha' : E' \xrightarrow[p]{r} E$ and $\alpha'' : E'' \xrightarrow[p]{r} E$ at a point p are *equivalent* if there exists an isomorphism $\phi : E' \to E''$ such that $\alpha' = \alpha'' \circ \phi$ i.e.

$$\begin{tikzcd}
0 \arrow[r] & E' \arrow[r, "\alpha'"] \arrow[d, "\cong\ \phi"'] & E \arrow[r] \arrow[d] & \mathcal{O}_p{}^{\oplus r} \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & E'' \arrow[r, "\alpha''"] & E \arrow[r] & \mathcal{O}_p{}^{\oplus r} \arrow[r] & 0
\end{tikzcd}$$

commutes.

- Denote by $\mathcal{H}_r(X, E, p)$ the set of equivalence classes of Hecke modifications of E of weight r at the point p.

Hall numbers

- We can consider more general short exact sequences

$$0 \to G \to E \to F \to 0$$

where G, E, F are simply coherent sheaves.

- Given a triple (G, E, F) of coherent sheaves on X, define *Hall number*

$$h^E_{F,G} := \frac{\#\{0 \to G \to E \to F \to 0\}}{\#\mathrm{Aut}(F)\#\mathrm{Aut}(G)}$$

- **Proposition** (R. Alvarenga): Let X be defined over a finite field. For a fixed $E \in \mathrm{Bun}_n X$, the Hecke modifications $[E' \to E]$ are completely determined (upto equivalence) by the hall number $h^E_{\mathcal{O}_p^{\oplus r}, E'}$, where E' runs through all $\mathrm{Bun}_n X$.

Hall numbers

- We can consider more general short exact sequences

$$0 \to G \to E \to F \to 0$$

where G, E, F are simply coherent sheaves.

- Given a triple (G, E, F) of coherent sheaves on X, define *Hall number*

$$h^{E}_{F,G} := \frac{\#\{0 \to G \to E \to F \to 0\}}{\#\mathrm{Aut}(F)\#\mathrm{Aut}(G)}$$

- **Proposition** (R. Alvarenga): Let X be defined over a finite field. For a fixed $E \in \mathrm{Bun}_n X$, the Hecke modifications $[E' \to E]$ are completely determined (upto equivalence) by the hall number $h^{E}_{\mathcal{O}_p^{\oplus r}, E'}$, where E' runs through all $\mathrm{Bun}_n X$.

Hall numbers

- We can consider more general short exact sequences

$$0 \to G \to E \to F \to 0$$

where G, E, F are simply coherent sheaves.

- Given a triple (G, E, F) of coherent sheaves on X, define *Hall number*

$$h^{E}_{F,G} := \frac{\#\{0 \to G \to E \to F \to 0\}}{\#\mathrm{Aut}(F)\#\mathrm{Aut}(G)}$$

- **Proposition** (R. Alvarenga): Let X be defined over a finite field. For a fixed $E \in \mathrm{Bun}_n X$, the Hecke modifications $[E' \to E]$ are completely determined (upto equivalence) by the hall number $h^{E}_{\mathcal{O}_p^{\oplus r}, E'}$, where E' runs through all $\mathrm{Bun}_n X$.

General motivation: Langlands correspondence

- One of the motivations to study Hecke modifications comes from Langlands program where one studies the action of 'Hecke operators' on 'unramified automorphic forms'.
- Let $\mathbb{F} = \mathbb{F}_q$ be the finite field with q elements, where q is a prime power.
- X geometrically irreducible smooth projective curve defined over $\mathbb{F}$ and denote by $\mathrm{Bun}_n X$ the set of isomorphism classes of rank n vector bundles over X.
- **Theorem (A. Weil)**: The classical (unramified) automorphic forms can be seen as complex valued functions (denoted AF_n) on $\mathrm{Bun}_n X$.

General motivation: Langlands correspondence

- One of the motivations to study Hecke modifications comes from Langlands program where one studies the action of 'Hecke operators' on 'unramified automorphic forms'.
- Let $\mathbb{F} = \mathbb{F}_q$ be the finite field with q elements, where q is a prime power.
- X geometrically irreducible smooth projective curve defined over $\mathbb{F}$ and denote by $\mathrm{Bun}_n X$ the set of isomorphism classes of rank n vector bundles over X.
- **Theorem (A. Weil)**: The classical (unramified) automorphic forms can be seen as complex valued functions (denoted AF_n) on $\mathrm{Bun}_n X$.

General motivation: Langlands correspondence

- One of the motivations to study Hecke modifications comes from Langlands program where one studies the action of 'Hecke operators' on 'unramified automorphic forms'.
- Let $\mathbb{F} = \mathbb{F}_q$ be the finite field with q elements, where q is a prime power.
- X geometrically irreducible smooth projective curve defined over $\mathbb{F}$ and denote by $\mathrm{Bun}_n X$ the set of isomorphism classes of rank n vector bundles over X.
- **Theorem (A. Weil)**: The classical (unramified) automorphic forms can be seen as complex valued functions (denoted AF_n) on $\mathrm{Bun}_n X$.

General motivation: Langlands correspondence

- One of the motivations to study Hecke modifications comes from Langlands program where one studies the action of 'Hecke operators' on 'unramified automorphic forms'.
- Let $\mathbb{F} = \mathbb{F}_q$ be the finite field with q elements, where q is a prime power.
- X geometrically irreducible smooth projective curve defined over $\mathbb{F}$ and denote by $\mathrm{Bun}_n X$ the set of isomorphism classes of rank n vector bundles over X.
- **Theorem (A. Weil)**: The classical (unramified) automorphic forms can be seen as complex valued functions (denoted AF_n) on $\mathrm{Bun}_n X$.

Langlands correspondence

- Let $F := \mathbb{F}_q(X)$ be the function field of X, $\mathbb{A}$ its adelic ring, $\mathcal{O}_{\mathbb{A}} \subset \mathbb{A}$ the ring of adelic integers.
- Weil's theorem can be stated more precisely as follows

$$\mathrm{Bun}_n X \xleftrightarrow{1:1} \mathrm{GL}_n F \setminus \mathrm{GL}_n \mathbb{A} / \mathrm{GL}_n \mathcal{O}_{\mathbb{A}}.$$

- For K a coherent sheaf on X (e.g. $K = \mathcal{O}_p{}^{\oplus r}$) we define the *Hecke operator*

$$T_K : \mathrm{AF}_n \to \mathrm{AF}_n$$

given by

$$(T_K f)(E) := \sum_{\substack{E' \subset E \\ E/E' \cong K}} f(E')$$

where $f \in \mathrm{AF}_n$ is a complex valued function (automorphic form) on $\mathrm{Bun}_n X$ and E' runs over the coherent subsheaves of E in $\mathrm{Bun}_n X$.
- These E' are the Hecke modifications of E!

Langlands correspondence

- Let $F := \mathbb{F}_q(X)$ be the function field of X, $\mathbb{A}$ its adelic ring, $\mathcal{O}_{\mathbb{A}} \subset \mathbb{A}$ the ring of adelic integers.
- Weil's theorem can be stated more precisely as follows

$$\mathrm{Bun}_n X \xleftrightarrow{1:1} \mathsf{GL}_n\, F \setminus \mathsf{GL}_n\, \mathbb{A} / \mathsf{GL}_n\, \mathcal{O}_{\mathbb{A}}.$$

- For K a coherent sheaf on X (e.g. $K = {\mathcal{O}_p}^{\oplus r}$) we define the *Hecke operator*

$$T_K : \mathrm{AF}_n \to \mathrm{AF}_n$$

given by

$$(T_K f)(E) := \sum_{\substack{E' \subset E \\ E/E' \cong K}} f(E')$$

where $f \in \mathrm{AF}_n$ is a complex valued function (automorphic form) on $\mathrm{Bun}_n X$ and E' runs over the coherent subsheaves of E in $\mathrm{Bun}_n X$.
- These E' are the Hecke modifications of E!

Langlands correspondence

- Let $F := \mathbb{F}_q(X)$ be the function field of X, $\mathbb{A}$ its adelic ring, $\mathcal{O}_{\mathbb{A}} \subset \mathbb{A}$ the ring of adelic integers.
- Weil's theorem can be stated more precisely as follows

$$\mathrm{Bun}_n X \stackrel{1:1}{\longleftrightarrow} \mathsf{GL}_n F \setminus \mathsf{GL}_n \mathbb{A} / \mathsf{GL}_n \mathcal{O}_{\mathbb{A}}.$$

- For K a coherent sheaf on X (e.g. $K = {\mathcal{O}_p}^{\oplus r}$) we define the *Hecke operator*

$$T_K : \mathrm{AF}_n \to \mathrm{AF}_n$$

given by

$$(T_K f)(E) := \sum_{\substack{E' \subset E \\ E/E' \cong K}} f(E')$$

where $f \in \mathrm{AF}_n$ is a complex valued function (automorphic form) on $\mathrm{Bun}_n X$ and E' runs over the coherent subsheaves of E in $\mathrm{Bun}_n X$.
- These E' are the Hecke modifications of E!

Langlands correspondence

- Let $F := \mathbb{F}_q(X)$ be the function field of X, $\mathbb{A}$ its adelic ring, $\mathcal{O}_{\mathbb{A}} \subset \mathbb{A}$ the ring of adelic integers.
- Weil's theorem can be stated more precisely as follows

$$\mathrm{Bun}_n X \stackrel{1:1}{\longleftrightarrow} \mathsf{GL}_n\, F \setminus \mathsf{GL}_n\, \mathbb{A} / \mathsf{GL}_n\, \mathcal{O}_{\mathbb{A}}.$$

- For K a coherent sheaf on X (e.g. $K = {\mathcal{O}_p}^{\oplus r}$) we define the *Hecke operator*

$$T_K : \mathrm{AF}_n \to \mathrm{AF}_n$$

given by

$$(T_K f)(E) := \sum_{\substack{E' \subset E \\ E/E' \cong K}} f(E')$$

where $f \in \mathrm{AF}_n$ is a complex valued function (automorphic form) on $\mathrm{Bun}_n X$ and E' runs over the coherent subsheaves of E in $\mathrm{Bun}_n X$.
- These E' are the Hecke modifications of E!

Explicit Hecke modifications in rank 2

- **Question**: Given a vector bundle E on X, can we explicitly say what are all the Hecke modifications of E?
- Obviously question makes sense only when we know what any vector bundle on the curve looks like!
- Recall, when $X = \mathbb{P}^1$, by Birkhoff-Grothendieck we know a vector bundle $E = \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(a_i)$ for a unique collection of $a_i \in \mathbb{Z}$.
- For the case of X an elliptic curve, there is a classification of vector bundles on X by Atiyah.
- D. Boozer (2020) used this to describe all possible Hecke modifications of all possible rank 2 bundles over $\mathbb{P}^1$ and an elliptic curve defined over $\mathbb{C}$.

Explicit Hecke modifications in rank 2

- **Question**: Given a vector bundle E on X, can we explicitly say what are all the Hecke modifications of E?
- Obviously question makes sense only when we know what any vector bundle on the curve looks like!
- Recall, when $X = \mathbb{P}^1$, by Birkhoff-Grothendieck we know a vector bundle $E = \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(a_i)$ for a unique collection of $a_i \in \mathbb{Z}$.
- For the case of X an elliptic curve, there is a classification of vector bundles on X by Atiyah.
- D. Boozer (2020) used this to describe all possible Hecke modifications of all possible rank 2 bundles over $\mathbb{P}^1$ and an elliptic curve defined over $\mathbb{C}$.

Explicit Hecke modifications in rank 2

- **Question**: Given a vector bundle E on X, can we explicitly say what are all the Hecke modifications of E?
- Obviously question makes sense only when we know what any vector bundle on the curve looks like!
- Recall, when $X = \mathbb{P}^1$, by Birkhoff-Grothendieck we know a vector bundle $E = \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(a_i)$ for a unique collection of $a_i \in \mathbb{Z}$.
- For the case of X an elliptic curve, there is a classification of vector bundles on X by Atiyah.
- D. Boozer (2020) used this to describe all possible Hecke modifications of all possible rank 2 bundles over $\mathbb{P}^1$ and an elliptic curve defined over $\mathbb{C}$.

Explicit Hecke modifications in rank 2

- **Question**: Given a vector bundle E on X, can we explicitly say what are all the Hecke modifications of E?
- Obviously question makes sense only when we know what any vector bundle on the curve looks like!
- Recall, when $X = \mathbb{P}^1$, by Birkhoff-Grothendieck we know a vector bundle $E = \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(a_i)$ for a unique collection of $a_i \in \mathbb{Z}$.
- For the case of X an elliptic curve, there is a classification of vector bundles on X by Atiyah.
- D. Boozer (2020) used this to describe all possible Hecke modifications of all possible rank 2 bundles over $\mathbb{P}^1$ and an elliptic curve defined over $\mathbb{C}$.

Explicit Hecke modifications in rank 2

- **Question**: Given a vector bundle E on X, can we explicitly say what are all the Hecke modifications of E?
- Obviously question makes sense only when we know what any vector bundle on the curve looks like!
- Recall, when $X = \mathbb{P}^1$, by Birkhoff-Grothendieck we know a vector bundle $E = \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(a_i)$ for a unique collection of $a_i \in \mathbb{Z}$.
- For the case of X an elliptic curve, there is a classification of vector bundles on X by Atiyah.
- D. Boozer (2020) used this to describe all possible Hecke modifications of all possible rank 2 bundles over $\mathbb{P}^1$ and an elliptic curve defined over $\mathbb{C}$.

Higher rank

- Let $X = \mathbb{P}^1$ defined over $\mathbb{C}$. David Boozer showed:
- For $m \geq 1$, the possible Hecke modifications of $\mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}$ are $\mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ and $\mathcal{O}_{\mathbb{P}^1}(m-1) \oplus \mathcal{O}_{\mathbb{P}^1}$.
- The only possible Hecke modification for $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ is $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.
- Roberto Alvarenga described the possible Hecke modifications for any rank n bundle on an elliptic curve defined over a finite field.
- The case of Hecke modifications for any rank n bundle on $\mathbb{P}^1$ over a finite field was worked on by my Phd student Leonardo Moco for his thesis.

Higher rank

- Let $X = \mathbb{P}^1$ defined over $\mathbb{C}$. David Boozer showed:
- For $m \geq 1$, the possible Hecke modifications of $\mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}$ are $\mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ and $\mathcal{O}_{\mathbb{P}^1}(m-1) \oplus \mathcal{O}_{\mathbb{P}^1}$.
- The only possible Hecke modification for $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ is $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.
- Roberto Alvarenga described the possible Hecke modifications for any rank n bundle on an elliptic curve defined over a finite field.
- The case of Hecke modifications for any rank n bundle on $\mathbb{P}^1$ over a finite field was worked on by my Phd student Leonardo Moco for his thesis.

Higher rank

- Let $X = \mathbb{P}^1$ defined over $\mathbb{C}$. David Boozer showed:
- For $m \geq 1$, the possible Hecke modifications of $\mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}$ are $\mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ and $\mathcal{O}_{\mathbb{P}^1}(m-1) \oplus \mathcal{O}_{\mathbb{P}^1}$.
- The only possible Hecke modification for $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ is $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.
- Roberto Alvarenga described the possible Hecke modifications for any rank n bundle on an elliptic curve defined over a finite field.
- The case of Hecke modifications for any rank n bundle on $\mathbb{P}^1$ over a finite field was worked on by my Phd student Leonardo Moco for his thesis.

Higher rank

- Let $X = \mathbb{P}^1$ defined over $\mathbb{C}$. David Boozer showed:
- For $m \geq 1$, the possible Hecke modifications of $\mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}$ are $\mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ and $\mathcal{O}_{\mathbb{P}^1}(m-1) \oplus \mathcal{O}_{\mathbb{P}^1}$.
- The only possible Hecke modification for $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ is $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.
- Roberto Alvarenga described the possible Hecke modifications for any rank n bundle on an elliptic curve defined over a finite field.
- The case of Hecke modifications for any rank n bundle on $\mathbb{P}^1$ over a finite field was worked on by my Phd student Leonardo Moco for his thesis.

Higher rank

- Let $X = \mathbb{P}^1$ defined over $\mathbb{C}$. David Boozer showed:
- For $m \geq 1$, the possible Hecke modifications of $\mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}$ are $\mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ and $\mathcal{O}_{\mathbb{P}^1}(m-1) \oplus \mathcal{O}_{\mathbb{P}^1}$.
- The only possible Hecke modification for $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ is $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.
- Roberto Alvarenga described the possible Hecke modifications for any rank n bundle on an elliptic curve defined over a finite field.
- The case of Hecke modifications for any rank n bundle on $\mathbb{P}^1$ over a finite field was worked on by my Phd student Leonardo Moco for his thesis.

Moduli space of Hecke modifications

- $\mathcal{H}_r(X, E, p)$ is not just a set but a moduli space!
- **Theorem:** $\mathcal{H}_r(X, E, p)$ is in bijection with the $k(p)$-rational points in the Grassmannian $\mathrm{Gr}(n-r, n)$, where $k(p)$ is the residue field at p.
- Over $\mathbb{C}$, rank $n = 2$, this is shown by D. Boozer.
- Case of arbitrary rank n vector bundle can be found in 'Introduction to Langlands program'.

Moduli space of Hecke modifications

- $\mathcal{H}_r(X, E, p)$ is not just a set but a moduli space!
- **Theorem:** $\mathcal{H}_r(X, E, p)$ is in bijection with the $k(p)$-rational points in the Grassmannian $\mathrm{Gr}(n-r, n)$, where $k(p)$ is the residue field at p.
- Over $\mathbb{C}$, rank $n = 2$, this is shown by D. Boozer.
- Case of arbitrary rank n vector bundle can be found in 'Introduction to Langlands program'.

Moduli space of Hecke modifications

- $\mathcal{H}_r(X, E, p)$ is not just a set but a moduli space!
- **Theorem:** $\mathcal{H}_r(X, E, p)$ is in bijection with the $k(p)$-rational points in the Grassmannian $\mathrm{Gr}(n-r, n)$, where $k(p)$ is the residue field at p.
- Over $\mathbb{C}$, rank $n = 2$, this is shown by D. Boozer.
- Case of arbitrary rank n vector bundle can be found in 'Introduction to Langlands program'.

Moduli space of Hecke modifications

- $\mathcal{H}_r(X, E, p)$ is not just a set but a moduli space!
- **Theorem:** $\mathcal{H}_r(X, E, p)$ is in bijection with the $k(p)$-rational points in the Grassmannian $\mathrm{Gr}(n-r, n)$, where $k(p)$ is the residue field at p.
- Over $\mathbb{C}$, rank $n = 2$, this is shown by D. Boozer.
- Case of arbitrary rank n vector bundle can be found in 'Introduction to Langlands program'.

Parabolic Bundles

- X smooth curve, $p_1, \dots p_s$ fixed set of s distinct closed points of X.
- A *parabolic vector bundle* of rank n at $p_1, \dots p_s$ is a vector bundle E on X of rank n together with:
- A *quasi-parabolic structure* i.e. a flag on E_{p_i} (the fiber of E at p_i) at each point p_i for $1 \leq i \leq s$

$$E_{p_i} = F_1(E_{p_i}) \supset F_2(E_{p_i}) \supset \cdots \supset F_n(E_{p_i})$$

where $F_j(E_{p_i})$ are subspaces of the vector space E_{p_i}.

- $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ with $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_{n-1} < \alpha_n < 1$ attached to $F_1(E_{p_i}), \dots, F_n(E_{p_i})$ are called *weights*.
- For $i = 1, \dots s$, $m_{1,i} = \dim F_1(E_{p_i}) - \dim F_2(E_{p_i}), \dots, m_{n,i} = \dim F_n(E_{p_i})$ are called the *multiplicities* of $\alpha_1, \dots \alpha_n$ at i.

Parabolic Bundles

- X smooth curve, $p_1, \ldots p_s$ fixed set of s distinct closed points of X.
- A *parabolic vector bundle* of rank n at $p_1, \ldots p_s$ is a vector bundle E on X of rank n together with:
- A *quasi-parabolic structure* i.e. a flag on E_{p_i} (the fiber of E at p_i) at each point p_i for $1 \leq i \leq s$

$$E_{p_i} = F_1(E_{p_i}) \supset F_2(E_{p_i}) \supset \cdots \supset F_n(E_{p_i})$$

where $F_j(E_{p_i})$ are subspaces of the vector space E_{p_i}.
- $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ with $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_{n-1} < \alpha_n < 1$ attached to $F_1(E_{p_i}), \ldots, F_n(E_{p_i})$ are called *weights*.
- For $i = 1, \ldots s$, $m_{1,i} = \dim F_1(E_{p_i}) - \dim F_2(E_{p_i}), \ldots, m_{n,i} = \dim F_n(E_{p_i})$ are called the *multiplicities* of $\alpha_1, \ldots \alpha_n$ at i.

Parabolic Bundles

- X smooth curve, $p_1, \ldots p_s$ fixed set of s distinct closed points of X.
- A *parabolic vector bundle* of rank n at $p_1, \ldots p_s$ is a vector bundle E on X of rank n together with:
- A *quasi-parabolic structure* i.e. a flag on E_{p_i} (the fiber of E at p_i) at each point p_i for $1 \leq i \leq s$

$$E_{p_i} = F_1(E_{p_i}) \supset F_2(E_{p_i}) \supset \cdots \supset F_n(E_{p_i})$$

where $F_j(E_{p_i})$ are subspaces of the vector space E_{p_i}.

- $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ with $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_{n-1} < \alpha_n < 1$ attached to $F_1(E_{p_i}), \ldots, F_n(E_{p_i})$ are called *weights*.
- For $i = 1, \ldots s$, $m_{1,i} = \dim F_1(E_{p_i}) - \dim F_2(E_{p_i}), \ldots, m_{n,i} = \dim F_n(E_{p_i})$ are called the *multiplicities* of $\alpha_1, \ldots \alpha_n$ at i.

Parabolic Bundles

- X smooth curve, $p_1, \dots p_s$ fixed set of s distinct closed points of X.
- A *parabolic vector bundle* of rank n at $p_1, \dots p_s$ is a vector bundle E on X of rank n together with:
- A *quasi-parabolic structure* i.e. a flag on E_{p_i} (the fiber of E at p_i) at each point p_i for $1 \leq i \leq s$

$$E_{p_i} = F_1(E_{p_i}) \supset F_2(E_{p_i}) \supset \cdots \supset F_n(E_{p_i})$$

where $F_j(E_{p_i})$ are subspaces of the vector space E_{p_i}.
- $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ with $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_{n-1} < \alpha_n < 1$ attached to $F_1(E_{p_i}), \dots, F_n(E_{p_i})$ are called *weights*.
- For $i = 1, \dots s$, $m_{1,i} = \dim F_1(E_{p_i}) - \dim F_2(E_{p_i}), \dots, m_{n,i} = \dim F_n(E_{p_i})$ are called the *multiplicities* of $\alpha_1, \dots \alpha_n$ at i.

Parabolic Bundles

- X smooth curve, $p_1, \dots p_s$ fixed set of s distinct closed points of X.
- A *parabolic vector bundle* of rank n at $p_1, \dots p_s$ is a vector bundle E on X of rank n together with:
- A *quasi-parabolic structure* i.e. a flag on E_{p_i} (the fiber of E at p_i) at each point p_i for $1 \leq i \leq s$

$$E_{p_i} = F_1(E_{p_i}) \supset F_2(E_{p_i}) \supset \cdots \supset F_n(E_{p_i})$$

where $F_j(E_{p_i})$ are subspaces of the vector space E_{p_i}.
- $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ with $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_{n-1} < \alpha_n < 1$ attached to $F_1(E_{p_i}), \dots, F_n(E_{p_i})$ are called *weights*.
- For $i = 1, \dots s$,
$m_{1,i} = \dim F_1(E_{p_i}) - \dim F_2(E_{p_i}), \dots, m_{n,i} = \dim F_n(E_{p_i})$
are called the *multiplicities* of $\alpha_1, \dots \alpha_n$ at i.

The moduli space of (semi)stable parabolic bundles

- E a parabolic bundle on X with parabolic weights $\alpha_{1,i}, \dots \alpha_{n,i}$ with multiplicities $m_{1,i}, \dots m_{n,i}$ for $i = 1, \dots s$.
- The *parabolic degree* of E is defined by

$$\mathrm{Pardeg}(E) = \deg(E) + \sum_i (\sum_j m_{j,i}\alpha_{j,i})$$

- A parabolic structure on a sub-bundle G of a parabolic vector bundle E is given as follows: for $p \in S$, the flag on G_p is induced by taking intersection with the flag of E_p and weight attached for the subspace $F_q(G_p)$ is $\beta_q = \alpha_i$ where i is the largest integer such that $F_q(G_p) \subset F_i(E_p)$.

The moduli space of (semi)stable parabolic bundles

- E a parabolic bundle on X with parabolic weights $\alpha_{1,i}, \ldots \alpha_{n,i}$ with multiplicities $m_{1,i}, \ldots m_{n,i}$ for $i = 1, \ldots s$.
- The *parabolic degree* of E is defined by

$$\mathrm{Pardeg}(E) = \deg(E) + \sum_i (\sum_j m_{j,i}\alpha_{j,i})$$

- A parabolic structure on a sub-bundle G of a parabolic vector bundle E is given as follows: for $p \in S$, the flag on G_p is induced by taking intersection with the flag of E_p and weight attached for the subspace $F_q(G_p)$ is $\beta_q = \alpha_i$ where i is the largest integer such that $F_q(G_p) \subset F_i(E_p)$.

The moduli space of (semi)stable parabolic bundles

- E a parabolic bundle on X with parabolic weights $\alpha_{1,i}, \ldots \alpha_{n,i}$ with multiplicities $m_{1,i}, \ldots m_{n,i}$ for $i = 1, \ldots s$.
- The *parabolic degree* of E is defined by

$$\mathrm{Pardeg}(E) = \deg(E) + \sum_i (\sum_j m_{j,i}\alpha_{j,i})$$

- A parabolic structure on a sub-bundle G of a parabolic vector bundle E is given as follows: for $p \in S$, the flag on G_p is induced by taking intersection with the flag of E_p and weight attached for the subspace $F_q(G_p)$ is $\beta_q = \alpha_i$ where i is the largest integer such that $F_q(G_p) \subset F_i(E_p)$.

Moduli space of parabolic bundles $\mathcal{M}_\alpha$

- A parabolic bundle E is *parabolic stable* if for every parabolic sub-bundle G of E

$$\frac{\mathrm{Pardeg}(G)}{rank(G)} < \frac{\mathrm{Pardeg}(E)}{rank(E)}$$

- *Parabolic semi-stable*: $\leq$.
- The weights are called *generic* if semi-stability and stability coincide.
- **Theorem**(Mehta-Seshadri): There exists a moduli space for semi-stable, rank n parabolic vector bundles, denoted $\mathcal{M}_\alpha$. Let the genus $g(X) \geq 2$.This is a normal projective variety of dimension $(g-1)n^2 + \sum_{p_i \in S} \frac{1}{2}(n^2 - \sum_j {m_j}^2)$.
- For generic weights, $\mathcal{M}_\alpha$ is smooth.

Moduli space of parabolic bundles $\mathcal{M}_\alpha$

- A parabolic bundle E is *parabolic stable* if for every parabolic sub-bundle G of E

$$\frac{\mathrm{Pardeg}(G)}{rank(G)} < \frac{\mathrm{Pardeg}(E)}{rank(E)}$$

- *Parabolic semi-stable*: $\leq$.
- The weights are called *generic* if semi-stability and stability coincide.
- **Theorem**(Mehta-Seshadri): There exists a moduli space for semi-stable, rank n parabolic vector bundles, denoted $\mathcal{M}_\alpha$. Let the genus $g(X) \geq 2$.This is a normal projective variety of dimension $(g-1)n^2 + \sum_{p_i \in S} \frac{1}{2}(n^2 - \sum_j {m_j}^2)$.
- For generic weights, $\mathcal{M}_\alpha$ is smooth.

Moduli space of parabolic bundles $\mathcal{M}_\alpha$

- A parabolic bundle E is *parabolic stable* if for every parabolic sub-bundle G of E

$$\frac{\mathrm{Pardeg}(G)}{rank(G)} < \frac{\mathrm{Pardeg}(E)}{rank(E)}$$

- *Parabolic semi-stable*: $\leq$.
- The weights are called *generic* if semi-stability and stability coincide.
- **Theorem**(Mehta-Seshadri): There exists a moduli space for semi-stable, rank n parabolic vector bundles, denoted $\mathcal{M}_\alpha$. Let the genus $g(X) \geq 2$.This is a normal projective variety of dimension $(g-1)n^2 + \sum_{p_i \in S} \frac{1}{2}(n^2 - \sum_j {m_j}^2)$.
- For generic weights, $\mathcal{M}_\alpha$ is smooth.

Moduli space of parabolic bundles $\mathcal{M}_\alpha$

- A parabolic bundle E is *parabolic stable* if for every parabolic sub-bundle G of E

$$\frac{\mathrm{Pardeg}(G)}{rank(G)} < \frac{\mathrm{Pardeg}(E)}{rank(E)}$$

- *Parabolic semi-stable*: $\leq$.
- The weights are called *generic* if semi-stability and stability coincide.
- **Theorem**(Mehta-Seshadri): There exists a moduli space for semi-stable, rank n parabolic vector bundles, denoted $\mathcal{M}_\alpha$. Let the genus $g(X) \geq 2$.This is a normal projective variety of dimension $(g-1)n^2 + \sum\limits_{p_i \in S} \frac{1}{2}(n^2 - \sum_j {m_j}^2)$.
- For generic weights, $\mathcal{M}_\alpha$ is smooth.

Moduli space of parabolic bundles $\mathcal{M}_\alpha$

- A parabolic bundle E is *parabolic stable* if for every parabolic sub-bundle G of E

$$\frac{\mathrm{Pardeg}(G)}{rank(G)} < \frac{\mathrm{Pardeg}(E)}{rank(E)}$$

- *Parabolic semi-stable*: $\leq$.
- The weights are called *generic* if semi-stability and stability coincide.
- **Theorem**(Mehta-Seshadri): There exists a moduli space for semi-stable, rank n parabolic vector bundles, denoted $\mathcal{M}_\alpha$. Let the genus $g(X) \geq 2$.This is a normal projective variety of dimension $(g-1)n^2 + \sum_{p_i \in S} \frac{1}{2}(n^2 - \sum_j {m_j}^2)$.
- For generic weights, $\mathcal{M}_\alpha$ is smooth.

Rank 2 correspondence

- Let $\mathcal{QP}(X, E, p)$ be the set of quasi-parabolic structures on X centred at the point p.
- Let $\mathcal{H}_r(X, E, p)$ be the set of Hecke modifications of weight 1 centred at the point p.
- There is a canonical isomorphism

$$\mathcal{H}_r(X, E, p) \to \mathcal{QP}(X, E, p)$$

- $[F \xrightarrow[p]{\alpha} E] \mapsto (E, \ell_p)$ where $\ell_p := \mathrm{im}(\alpha : F_p \to E_p)$.
- Given a line ℓ_p we construct a Hecke modification $[F \xrightarrow{\alpha} E]$, by setting F as the sheaf corresponding to the subsheaf $\mathcal{F} \subset \mathcal{E}$ of the sheaf of sections of E defined by

$$\mathcal{F}(U) := \{s \in \mathcal{E}(U) |\ s(p) \in \ell_p\}.$$

for $U \subset X$ open. Then $[F \xrightarrow{\alpha} E]$ corresponds to ℓ_p.

Rank 2 correspondence

- Let $\mathcal{QP}(X, E, p)$ be the set of quasi-parabolic structures on X centred at the point p.
- Let $\mathcal{H}_r(X, E, p)$ be the set of Hecke modifications of weight 1 centred at the point p.
- There is a canonical isomorphism

$$\mathcal{H}_r(X, E, p) \to \mathcal{QP}(X, E, p)$$

- $[F \xrightarrow[p]{\alpha} E] \mapsto (E, \ell_p)$ where $\ell_p := \mathrm{im}(\alpha : F_p \to E_p)$.
- Given a line ℓ_p we construct a Hecke modification $[F \xrightarrow{\alpha} E]$, by setting F as the sheaf corresponding to the subsheaf $\mathcal{F} \subset \mathcal{E}$ of the sheaf of sections of E defined by

$$\mathcal{F}(U) := \{s \in \mathcal{E}(U) |\ s(p) \in \ell_p\}.$$

for $U \subset X$ open. Then $[F \xrightarrow{\alpha} E]$ corresponds to ℓ_p.

Rank 2 correspondence

- Let $\mathcal{QP}(X, E, p)$ be the set of quasi-parabolic structures on X centred at the point p.
- Let $\mathcal{H}_r(X, E, p)$ be the set of Hecke modifications of weight 1 centred at the point p.
- There is a canonical isomorphism

$$\mathcal{H}_r(X, E, p) \to \mathcal{QP}(X, E, p)$$

- $[F \xrightarrow[p]{\alpha} E] \mapsto (E, \ell_p)$ where $\ell_p := \mathrm{im}(\alpha : F_p \to E_p)$.
- Given a line ℓ_p we construct a Hecke modification $[F \xrightarrow{\alpha} E]$, by setting F as the sheaf corresponding to the subsheaf $\mathcal{F} \subset \mathcal{E}$ of the sheaf of sections of E defined by

$$\mathcal{F}(U) := \{s \in \mathcal{E}(U) | \ s(p) \in \ell_p\}.$$

for $U \subset X$ open. Then $[F \xrightarrow{\alpha} E]$ corresponds to ℓ_p.

Rank 2 correspondence

- Let $\mathcal{QP}(X, E, p)$ be the set of quasi-parabolic structures on X centred at the point p.
- Let $\mathcal{H}_r(X, E, p)$ be the set of Hecke modifications of weight 1 centred at the point p.
- There is a canonical isomorphism

$$\mathcal{H}_r(X, E, p) \to \mathcal{QP}(X, E, p)$$

- $[F \xrightarrow[p]{\alpha} E] \mapsto (E, \ell_p)$ where $\ell_p := \mathrm{im}(\alpha : F_p \to E_p)$.
- Given a line ℓ_p we construct a Hecke modification $[F \xrightarrow{\alpha} E]$, by setting F as the sheaf corresponding to the subsheaf $\mathcal{F} \subset \mathcal{E}$ of the sheaf of sections of E defined by

$$\mathcal{F}(U) := \{s \in \mathcal{E}(U) |\ s(p) \in \ell_p\}.$$

for $U \subset X$ open. Then $[F \xrightarrow{\alpha} E]$ corresponds to ℓ_p.

Rank 2 correspondence

- Let $\mathcal{QP}(X, E, p)$ be the set of quasi-parabolic structures on X centred at the point p.
- Let $\mathcal{H}_r(X, E, p)$ be the set of Hecke modifications of weight 1 centred at the point p.
- There is a canonical isomorphism

$$\mathcal{H}_r(X, E, p) \to \mathcal{QP}(X, E, p)$$

- $[F \xrightarrow[p]{\alpha} E] \mapsto (E, \ell_p)$ where $\ell_p := \mathrm{im}(\alpha : F_p \to E_p)$.
- Given a line ℓ_p we construct a Hecke modification $[F \xrightarrow{\alpha} E]$, by setting F as the sheaf corresponding to the subsheaf $\mathcal{F} \subset \mathcal{E}$ of the sheaf of sections of E defined by

$$\mathcal{F}(U) := \{s \in \mathcal{E}(U) |\ s(p) \in \ell_p\}.$$

for $U \subset X$ open. Then $[F \xrightarrow{\alpha} E]$ corresponds to ℓ_p.

Rank 2 correspondence between Hecke modifications and Quasi-parabolic structures

- We can also do this for several points.
- Let X be a smooth curve, $p_1, ..., p_s$ be s distinct points on X and E be a rank 2 vector bundle on X.
- Let $\mathcal{H}_r(X, E, p_1, ..., p_s)$ be the set of Hecke modifications of weight 1 centred at the points $p_1, ..., p_s$.
- **Theorem:** There is a canonical isomorphism $\mathcal{H}_r(X, E, p_1, ..., p_s) \to \mathcal{QP}(X, E, p_1, ..., p_s)$ given by

$$[E_s \xrightarrow[p_s]{\alpha_s} E_{s-1} \xrightarrow[p_{s-1}]{\alpha_{s-1}} \dots \xrightarrow[p_1]{\alpha_1} E] \mapsto (E, \ell_{p_1}, ..., \ell_{p_s})$$

  where $\ell_{p_i} := \mathrm{im}((\alpha_1 \circ \alpha_2 \circ \dots \circ \alpha_i)_{p_i} : (E_i)_{p_i} \to E_{p_i})$.
- The above is true as moduli spaces.

Rank 2 correspondence between Hecke modifications and Quasi-parabolic structures

- We can also do this for several points.
- Let X be a smooth curve, $p_1, \dots, p_s$ be s distinct points on X and E be a rank 2 vector bundle on X.
- Let $\mathcal{H}_r(X, E, p_1, \dots, p_s)$ be the set of Hecke modifications of weight 1 centred at the points $p_1, \dots, p_s$.
- **Theorem:** There is a canonical isomorphism $\mathcal{H}_r(X, E, p_1, \dots, p_s) \to \mathcal{QP}(X, E, p_1, \dots, p_s)$ given by

$$[E_s \xrightarrow[p_s]{\alpha_s} E_{s-1} \xrightarrow[p_{s-1}]{\alpha_{s-1}} \dots \xrightarrow[p_1]{\alpha_1} E] \mapsto (E, \ell_{p_1}, \dots, \ell_{p_s})$$

where $\ell_{p_i} := \mathrm{im}((\alpha_1 \circ \alpha_2 \circ \dots \circ \alpha_i)_{p_i} : (E_i)_{p_i} \to E_{p_i})$.
- The above is true as moduli spaces.

Rank 2 correspondence between Hecke modifications and Quasi-parabolic structures

- We can also do this for several points.
- Let X be a smooth curve, $p_1, ..., p_s$ be s distinct points on X and E be a rank 2 vector bundle on X.
- Let $\mathcal{H}_r(X, E, p_1, ..., p_s)$ be the set of Hecke modifications of weight 1 centred at the points $p_1, ..., p_s$.
- **Theorem:** There is a canonical isomorphism $\mathcal{H}_r(X, E, p_1, ..., p_s) \to \mathcal{QP}(X, E, p_1, ..., p_s)$ given by

$$[E_s \xrightarrow[p_s]{\alpha_s} E_{s-1} \xrightarrow[p_{s-1}]{\alpha_{s-1}} \dots \xrightarrow[p_1]{\alpha_1} E] \mapsto (E, \ell_{p_1}, ..., \ell_{p_s})$$

  where $\ell_{p_i} := \mathrm{im}((\alpha_1 \circ \alpha_2 \circ \dots \circ \alpha_i)_{p_i} : (E_i)_{p_i} \to E_{p_i})$.
- The above is true as moduli spaces.

Rank 2 correspondence between Hecke modifications and Quasi-parabolic structures

- We can also do this for several points.
- Let X be a smooth curve, $p_1, \ldots, p_s$ be s distinct points on X and E be a rank 2 vector bundle on X.
- Let $\mathcal{H}_r(X, E, p_1, \ldots, p_s)$ be the set of Hecke modifications of weight 1 centred at the points $p_1, \ldots, p_s$.
- **Theorem:** There is a canonical isomorphism $\mathcal{H}_r(X, E, p_1, \ldots, p_s) \to \mathcal{QP}(X, E, p_1, \ldots, p_s)$ given by

$$[E_s \xrightarrow[p_s]{\alpha_s} E_{s-1} \xrightarrow[p_{s-1}]{\alpha_{s-1}} \ldots \xrightarrow[p_1]{\alpha_1} E] \mapsto (E, \ell_{p_1}, \ldots, \ell_{p_s})$$

where $\ell_{p_i} := \mathrm{im}((\alpha_1 \circ \alpha_2 \circ \ldots \circ \alpha_i)_{p_i} : (E_i)_{p_i} \to E_{p_i})$.

- The above is true as moduli spaces.

Rank 2 correspondence between Hecke modifications and Quasi-parabolic structures

- We can also do this for several points.
- Let X be a smooth curve, $p_1, \ldots, p_s$ be s distinct points on X and E be a rank 2 vector bundle on X.
- Let $\mathcal{H}_r(X, E, p_1, \ldots, p_s)$ be the set of Hecke modifications of weight 1 centred at the points $p_1, \ldots, p_s$.
- **Theorem:** There is a canonical isomorphism $\mathcal{H}_r(X, E, p_1, \ldots, p_s) \to \mathcal{QP}(X, E, p_1, \ldots, p_s)$ given by

$$[E_s \xrightarrow[p_s]{\alpha_s} E_{s-1} \xrightarrow[p_{s-1}]{\alpha_{s-1}} \ldots \xrightarrow[p_1]{\alpha_1} E] \mapsto (E, \ell_{p_1}, \ldots, \ell_{p_s})$$

where $\ell_{p_i} := \mathrm{im}((\alpha_1 \circ \alpha_2 \circ \ldots \circ \alpha_i)_{p_i} : (E_i)_{p_i} \to E_{p_i})$.
- The above is true as moduli spaces.

Why care about this correspondence?

- Hecke modifications have applications to Langlands program but not much is known about simultaneous Hecke modifications of higher rank at several points.
- Moduli spaces of parabolic bundles are examples of varieties, application of GIT.
- Can also be helpful in understanding other varieties, e.g: let $s \geq 5$. The blow-up of $\mathbb{P}^s$ in $s+3$ points is isomorphic in codimension 1 to the moduli space of rank 2, trivial determinant, stable parabolic bundles on $\mathbb{P}^1$ with s parabolic points and weights $(\frac{1}{2}, \frac{1}{2}, \ldots \frac{1}{2})$.
- Our goal is to generalise the above correspondence for arbitrary rank any number of points and use it to study problems on either side.

Why care about this correspondence?

- Hecke modifications have applications to Langlands program but not much is known about simultaneous Hecke modifications of higher rank at several points.
- Moduli spaces of parabolic bundles are examples of varieties, application of GIT.
- Can also be helpful in understanding other varieties, e.g: let $s \geq 5$. The blow-up of $\mathbb{P}^s$ in $s+3$ points is isomorphic in codimension 1 to the moduli space of rank 2, trivial determinant, stable parabolic bundles on $\mathbb{P}^1$ with s parabolic points and weights $(\frac{1}{2}, \frac{1}{2}, \dots \frac{1}{2})$.
- Our goal is to generalise the above correspondence for arbitrary rank any number of points and use it to study problems on either side.

Why care about this correspondence?

- Hecke modifications have applications to Langlands program but not much is known about simultaneous Hecke modifications of higher rank at several points.
- Moduli spaces of parabolic bundles are examples of varieties, application of GIT.
- Can also be helpful in understanding other varieties, e.g: let $s \geq 5$. The blow-up of $\mathbb{P}^s$ in $s+3$ points is isomorphic in codimension 1 to the moduli space of rank 2, trivial determinant, stable parabolic bundles on $\mathbb{P}^1$ with s parabolic points and weights $(\frac{1}{2}, \frac{1}{2}, \ldots \frac{1}{2})$.
- Our goal is to generalise the above correspondence for arbitrary rank any number of points and use it to study problems on either side.

Why care about this correspondence?

- Hecke modifications have applications to Langlands program but not much is known about simultaneous Hecke modifications of higher rank at several points.
- Moduli spaces of parabolic bundles are examples of varieties, application of GIT.
- Can also be helpful in understanding other varieties, e.g: let $s \geq 5$. The blow-up of $\mathbb{P}^s$ in $s+3$ points is isomorphic in codimension 1 to the moduli space of rank 2, trivial determinant, stable parabolic bundles on $\mathbb{P}^1$ with s parabolic points and weights $(\frac{1}{2}, \frac{1}{2}, \dots \frac{1}{2})$.
- Our goal is to generalise the above correspondence for arbitrary rank any number of points and use it to study problems on either side.

Sequences of Hecke modifications for higher rank

- The first step is to study sequences of Hecke modifications for higher rank at several points.
- X a smooth curve, p a point of X and E a rank n vector bundle on X.
- A *sequence of Hecke modifications* of E at p with weights $r_i \in \mathbb{Z}_{\geq 1}$ is a collection of rank n vector bundles E_i and Hecke modifications $E_i \xrightarrow[r_i]{p} E_{i-1}$ for $i = 1, 2, ..., l$, denoted as

$$[E_\ell \xrightarrow[p]{r_\ell} E_{\ell-1} \to \cdots \to E_1 \xrightarrow[p]{r_1} E].$$

- We say the sequence is **complete** if $\ell = n$ and $r_i = 1$ for all $i = 1, \ldots, \ell$.

Sequences of Hecke modifications for higher rank

- The first step is to study sequences of Hecke modifications for higher rank at several points.
- X a smooth curve, p a point of X and E a rank n vector bundle on X.
- A *sequence of Hecke modifications* of E at p with weights $r_i \in \mathbb{Z}_{\geq 1}$ is a collection of rank n vector bundles E_i and Hecke modifications $E_i \xrightarrow[r_i]{p} E_{i-1}$ for $i = 1, 2, ..., l$, denoted as

$$[E_\ell \xrightarrow[p]{r_\ell} E_{\ell-1} \to \cdots \to E_1 \xrightarrow[p]{r_1} E].$$

- We say the sequence is **complete** if $\ell = n$ and $r_i = 1$ for all $i = 1, \ldots, \ell$.

Sequences of Hecke modifications for higher rank

- The first step is to study sequences of Hecke modifications for higher rank at several points.
- X a smooth curve, p a point of X and E a rank n vector bundle on X.
- A *sequence of Hecke modifications* of E at p with weights $r_i \in \mathbb{Z}_{\geq 1}$ is a collection of rank n vector bundles E_i and Hecke modifications $E_i \xrightarrow[r_i]{p} E_{i-1}$ for $i = 1, 2, ..., l$, denoted as

$$[E_\ell \xrightarrow[p]{r_\ell} E_{\ell-1} \to \cdots \to E_1 \xrightarrow[p]{r_1} E].$$

- We say the sequence is **complete** if $\ell = n$ and $r_i = 1$ for all $i = 1, \ldots, \ell$.

Sequences of Hecke modifications for higher rank

- The first step is to study sequences of Hecke modifications for higher rank at several points.
- X a smooth curve, p a point of X and E a rank n vector bundle on X.
- A *sequence of Hecke modifications* of E at p with weights $r_i \in \mathbb{Z}_{\geq 1}$ is a collection of rank n vector bundles E_i and Hecke modifications $E_i \xrightarrow[r_i]{p} E_{i-1}$ for $i = 1, 2, ..., l$, denoted as

$$[E_\ell \xrightarrow[p]{r_\ell} E_{\ell-1} \to \cdots \to E_1 \xrightarrow[p]{r_1} E].$$

- We say the sequence is **complete** if $\ell = n$ and $r_i = 1$ for all $i = 1, \ldots, \ell$.

Sequences and sets of Hecke modifications and flags of vector spaces

Proposition: X a smooth curve, p a point in X. Given $E \in \mathrm{Bun_n}(X)$. T.f.a.e:

- There is a flag of vector spaces of the fibre E_p
$$\{0\} = V_1 \subset V_2 \subset \cdots \subset V_\ell = E_p.$$
- There is a set of Hecke modifications of E at p with different decreasing weights
$$\{[E_i \xrightarrow[p]{r_i} E]; i = 1, \ldots, \ell,\ E_1 = E,\ E_\ell = E(-p)\}$$
and morphisms of coherent sheaves $\varphi_i : E_i \to E_{i-1}$ for all $i = 1, \ldots, \ell$, such that the Hecke modification $[E_i \xrightarrow[p]{r_i} E]$ factors through φ_i.
- There is a sequence of Hecke modifications of E at p
$$[E(-p) = E_\ell \xrightarrow[p]{r_{\ell-1}} E_{\ell-1} \xrightarrow[p]{r_{\ell-2}} \cdots \to E_1 \xrightarrow[p]{r_1} E].$$

Hecke modifications at several points

- Let $[\varphi_1 : E_1 \xrightarrow[p_1]{r_1} E]$ and $[\varphi_2 : E_2 \xrightarrow[p_2]{r_2} E]$ be Hecke modifications of E at points p_1 and p_2 respectively. The fibre product $E_1 \times_E E_2$ is a vector bundle with projection maps $[\mathrm{pr}_1 : E_1 \times_E E_2 \to E_1]$, $[\mathrm{pr}_2 : E_1 \times_E E_2 \to E_2]$ and $\psi := \varphi_1 \circ \mathrm{pr}_1 = \varphi_2 \circ \mathrm{pr}_2$ realizes Hecke modifications $[E_1 \times_E E_2 \xrightarrow[p_2]{r_2} E_1], [E_1 \times_E E_2 \xrightarrow[p_1]{r_1} E_2]$ and
$$[E_1 \times_E E_2 \xrightarrow[(p_1,p_2)]{(r_1,r_2)} E]$$ respectively.
- **Theorem:** (Alvarenga, Kaur, Moco): Suppose $D = p_1 + \cdots + p_s$ is a divisor of points on X and E a rank n vector bundle. Then, there is a correspondence between complete flags at D in E and sequences of simultaneous Hecke modifications of E at D of the form
$$[E(-D) \xrightarrow{D} E_{n-1} \xrightarrow{D} \cdots \xrightarrow{D} E_1 \xrightarrow{D} E].$$

Hecke modifications at several points

- Let $[\varphi_1 : E_1 \xrightarrow[p_1]{r_1} E]$ and $[\varphi_2 : E_2 \xrightarrow[p_2]{r_2} E]$ be Hecke modifications of E at points p_1 and p_2 respectively. The fibre product $E_1 \times_E E_2$ is a vector bundle with projection maps $[\mathrm{pr}_1 : E_1 \times_E E_2 \to E_1]$, $[\mathrm{pr}_2 : E_1 \times_E E_2 \to E_2]$ and $\psi := \varphi_1 \circ \mathrm{pr}_1 = \varphi_2 \circ \mathrm{pr}_2$ realizes Hecke modifications $[E_1 \times_E E_2 \xrightarrow[p_2]{r_2} E_1], [E_1 \times_E E_2 \xrightarrow[p_1]{r_1} E_2]$ and
$$[E_1 \times_E E_2 \xrightarrow[(p_1,p_2)]{(r_1,r_2)} E] \text{ respectively.}$$
- **Theorem:** (Alvarenga, Kaur, Moco): Suppose $D = p_1 + \cdots + p_s$ is a divisor of points on X and E a rank n vector bundle. Then, there is a correspondence between complete flags at D in E and sequences of simultaneous Hecke modifications of E at D of the form
$$[E(-D) \xrightarrow{D} E_{n-1} \xrightarrow{D} \cdots \xrightarrow{D} E_1 \xrightarrow{D} E].$$

The Hecke stack

- We follow Gaitsgory in defining the Hecke stack:

  Definition: Let $\ell, n \in \mathbb{N}$ and $p \in X$ be a point and k a field. The *Hecke stack* $\mathcal{H}_n^{(p,\ell)}$ is the sheaf of groupoids which assigns to a k-scheme T, the groupoid $\mathcal{H}_n^{(p,\ell)}(T)$ classifying the following data:

  - a sequence $(\mathcal{E}_0, \ldots, \mathcal{E}_\ell) \in \mathrm{Bun_n}(X \times T)$; and
  - embeddings $\mathcal{E}_i \hookrightarrow \mathcal{E}_{i+1}$ of coherent sheaves such that $\mathcal{E}_{i+1}/\mathcal{E}_i \in \mathrm{Bun_n}(\{p\} \times T)$.

  We can generalize this by varying p through the points of X. Consider $p : T \to X$ to be a morphism of schemes and require $\mathcal{E}_{i+1}/\mathcal{E}_i \in \mathrm{Bun_n}(\Gamma_p)$, where Γ_p is the graph of p. More generally, we consider a sequence $(p_1, \ldots, p_\ell)$ of morphism $p_i : T \to X$ such that $\mathcal{E}_{i+1}/\mathcal{E}_i \in \mathrm{Bun_n}(\Gamma_{p_{i+1}})$, where $\Gamma_{p_{i+1}}$ is the graph of p_{i+1}.

The Parabolic stack

- On the other hand we have the Parabolic stack:
- **Definition:** Suppose $D = \{p_1, \ldots, p_s\}$ a collection of distinct points at X, $\ell := (\ell_1, \ldots, \ell_s) \in \mathbb{N}^s$ and $\vec{n} = (n_{i,j}) \in \mathbb{N}^{\ell_1 + \cdots + \ell_s}$ be such that $n_{i,0} = n$ for every $i = 1, \ldots, s$, and $n_{i,0} \geq n_{i,1} \geq \cdots \geq n_{i,\ell_i}$ for each i fixed.
- We define the stack of parabolic bundles at X, of parabolic type (D, ℓ) and dimension vector $\vec{n}$, to be the stack $\mathcal{P}_{\vec{n}}^{(D,\ell)}$, which assigns to a k-scheme T, the groupoid $\mathcal{P}_{\vec{n}}^{(D,\ell)}(T)$ classifying the following data:
  - $\mathcal{E} \in \mathrm{Bun_n}(X \times T)$;
  - $\mathcal{E}|_{\{p_i\} \times T} = E_{i,0} \supset E_{i,1} \supset \cdots \supset E_{i,\ell_i} = 0$ is a filtration by vector bundles
  - $\mathrm{rk}(E_{i,j}) = n_{i,j}$.

The Parabolic stack

- On the other hand we have the Parabolic stack:
- **Definition:** Suppose $D = \{p_1, \ldots, p_s\}$ a collection of distinct points at X, $\ell := (\ell_1, \ldots, \ell_s) \in \mathbb{N}^s$ and $\vec{n} = (n_{i,j}) \in \mathbb{N}^{\ell_1 + \cdots + \ell_s}$ be such that $n_{i,0} = n$ for every $i = 1, \ldots, s$, and $n_{i,0} \geq n_{i,1} \geq \cdots \geq n_{i,\ell_i}$ for each i fixed.
- We define the stack of parabolic bundles at X, of parabolic type (D, ℓ) and dimension vector $\vec{n}$, to be the stack $\mathcal{P}_{\vec{n}}^{(D,\ell)}$, which assigns to a k-scheme T, the groupoid $\mathcal{P}_{\vec{n}}^{(D,\ell)}(T)$ classifying the following data:
  - $\mathcal{E} \in \mathrm{Bun_n}(X \times T)$;
  - $\mathcal{E}|_{\{p_i\} \times T} = E_{i,0} \supset E_{i,1} \supset \cdots \supset E_{i,\ell_i} = 0$ is a filtration by vector bundles
  - $\mathrm{rk}(E_{i,j}) = n_{i,j}$.

The Parabolic stack

- On the other hand we have the Parabolic stack:
- **Definition:** Suppose $D = \{p_1, \ldots, p_s\}$ a collection of distinct points at X, $\ell := (\ell_1, \ldots, \ell_s) \in \mathbb{N}^s$ and $\vec{n} = (n_{i,j}) \in \mathbb{N}^{\ell_1 + \cdots + \ell_s}$ be such that $n_{i,0} = n$ for every $i = 1, \ldots, s$, and $n_{i,0} \geq n_{i,1} \geq \cdots \geq n_{i,\ell_i}$ for each i fixed.
- We define the stack of parabolic bundles at X, of parabolic type (D, ℓ) and dimension vector $\vec{n}$, to be the stack $\mathcal{P}_{\vec{n}}^{(D,\ell)}$, which assigns to a k-scheme T, the groupoid $\mathcal{P}_{\vec{n}}^{(D,\ell)}(T)$ classifying the following data:
  - $\mathcal{E} \in \mathrm{Bun_n}(X \times T)$;
  - $\mathcal{E}|_{\{p_i\} \times T} = E_{i,0} \supset E_{i,1} \supset \cdots \supset E_{i,\ell_i} = 0$ is a filtration by vector bundles
  - $\mathrm{rk}(E_{i,j}) = n_{i,j}$.

Correspondence for stacks

- We can now state our correspondence for stacks.

  Theorem (Alvarenga, Kaur, Moco): Let $p \in X$ be a point, $\ell \in \mathbb{N}$ and $\vec{n} = (n_0, \ldots, n_\ell) \in \mathbb{N}^\ell$ such that $n_0 \geq n_1 \geq \cdots \geq n_\ell$. There exists a natural transformation

$$\eta : \mathcal{P}_{\vec{n}}^{(p,\ell)} \longrightarrow \mathcal{H}_n^{(p,\ell)}$$

where $n = n_0$, given as follows: for each k-scheme T,

$$\eta(T) : \mathcal{P}_{\vec{n}}^{(p,\ell)}(T) \longrightarrow \mathcal{H}_n^{(p,\ell)}(T)$$

Correspondence for stacks

$\eta(T)$ sends the data:

- $\mathcal{E} \in \mathrm{Bun_n}(X \times T)$;
- $\mathcal{E}|_{\{p\}\times T} = E_0 \supset E_1 \supset \cdots \supset E_\ell = 0$ is a filtration by vector bundles
- $\mathrm{rk}(E_i) = n_i$.

to

- the sequence $(\mathcal{E}_0, \ldots, \mathcal{E}_\ell) \in \mathrm{Bun_n}(X \times T)$, where $\mathcal{E}_\ell := \mathcal{E}$ and $\mathcal{E}_i$ is defined as follows: for $U \subseteq X \times T$ an open subset, $\mathcal{E}_i(U) := \{\sigma \in \mathcal{E}(U) \mid \text{ if } p \times T \subseteq U \text{ then } \sigma|_{p\times T} \subseteq E_{\ell-i}\}$ and
- inclusions $\mathcal{E}_i \hookrightarrow \mathcal{E}_{i+1}$ of coherent sheaves given by the restrictions.

Open questions

- Can we have this correspondence in terms of moduli spaces (locally finite schemes)?
  - What would be appropriate semistability conditions?
  - If we can construct such a moduli space, what is its geometry? i.e is it irreducible? how many components? is it smooth?
- Can we recover the curve from the moduli space of Hecke modifications i.e. have a Torelli-type result?
- Is there a dictionary with Higgs bundles?
- Do Hecke modifications in higher rank have a geometric meaning?

Open questions

- Can we have this correspondence in terms of moduli spaces (locally finite schemes)?
  - What would be appropriate semistability conditions?
  - If we can construct such a moduli space, what is its geometry? i.e is it irreducible? how many components? is it smooth?
- Can we recover the curve from the moduli space of Hecke modifications i.e. have a Torelli-type result?
- Is there a dictionary with Higgs bundles?
- Do Hecke modifications in higher rank have a geometric meaning?

Open questions

- Can we have this correspondence in terms of moduli spaces (locally finite schemes)?
  - What would be appropriate semistability conditions?
  - If we can construct such a moduli space, what is its geometry? i.e is it irreducible? how many components? is it smooth?
- Can we recover the curve from the moduli space of Hecke modifications i.e. have a Torelli-type result?
- Is there a dictionary with Higgs bundles?
- Do Hecke modifications in higher rank have a geometric meaning?

Open questions

- Can we have this correspondence in terms of moduli spaces (locally finite schemes)?
    - What would be appropriate semistability conditions?
    - If we can construct such a moduli space, what is its geometry? i.e is it irreducible? how many components? is it smooth?
- Can we recover the curve from the moduli space of Hecke modifications i.e. have a Torelli-type result?
- Is there a dictionary with Higgs bundles?
- Do Hecke modifications in higher rank have a geometric meaning?

Thank you for your attention !