Double Affine Demazure Products & Affine Quantum Bruhat Graphs

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The Demazure Product

Coxeter Definition

Let W be a Coxeter group. We define the **Demazure product** $*$ in W by:

(i) $x * y = xy$ if $\ell(x) + \ell(y) = \ell(xy)$.

(ii) $x * s = x$ if $\ell(xs) < \ell(x)$ for s a simple reflection.

Hecke Algebra Interpretation

Let $\mathcal{H}_0$ be the $q = 0$ specialisation of the Hecke algebra, generated by $\{T_w \mid w \in W\}$. Then:

$$T_x T_y = (-1)^{\ell(x)+\ell(y)-\ell(x*y)} T_{x*y}$$

The Demazure product has applications in many areas, such as affine Deligne-Lusztig varieties, but particularly as a tool in describing multiplication in Hecke algebras.

A relatively new object of interest is the **Kac-Moody affine Hecke algebra** $\widehat{\mathcal{H}}$, defined by Braverman, Kazhdan, Patnaik for untwisted affine and generalised all types by Bardy-Panse, Gaussent, Rousseau.

$\widehat{\mathcal{H}}$ is indexed by the double affine Weyl semigroup $W_{\mathcal{T}}$.

Remark

$W_{\mathcal{T}}$ is not a Coxeter group.

As a result, $W_{\mathcal{T}}$ has no natural Demazure product.

Goal: Define a **double affine Demazure product** for $W_{\mathcal{T}}$.

Much work has been done (Braverman, Kazhdan, Patnaik, and Muthiah, Orr) to give $W_{\mathcal{T}}$ Coxeter-like structures.

A conjecture due to Muthiah, Puskás gives us a candidate for a Demazure product in the double affine case by considering the Kac-Moody affine Hecke algebra directly. This direction could work, however we take a different approach.

We will look at work by Schremmer to find a definition for the Demazure product.

Ultimately, we want to reconnect this with the Hecke algebra.

Hecke Algebra Background

Let G be a split, simple Lie group over a field k, with Borel B. The *Bruhat decomposition* gives a bijection $B\backslash G/B \longleftrightarrow W$, where W is the Weyl group of G.

Example

Let $G = \mathrm{SL}_n$. Then B is the set of upper triangular matrices, and $B\backslash G/B$ is indexed by permutation matrices $\longrightarrow W = S_n$.

Let $L = k((t))$ be the field of formal Laurent series in t, with ring of integers $\mathcal{O} = k[[t]]$. The **Iwahori subgroup** I of G is

$$I = \{g \in G(\mathcal{O}) \mid g \in B(k) \text{ mod } t\}.$$

The *Cartan decomposition* gives a bijection $I\backslash G(L)/I \longleftrightarrow W_{\text{aff}}$, where W_{aff} is the affine Weyl group.

$W_{\text{aff}} \cong W \ltimes Q^\vee$, where $Q^\vee$ is the coroot lattice.

The **Iwahori-Hecke algebra** $\mathcal{H}$ for G is the convolution algebra of $\mathbb{C}$-valued functions on $G(L)$ which are I-bi-invariant.

$\mathcal{H}$ has basis $\{T_w \mid w \in W_{\text{aff}}\}$, where T_w represents the indicator function for the double coset IwI, subject to the relations:

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w), \\ qT_{sw} + (q-1)T_w & \text{if } \ell(sw) < \ell(w). \end{cases}$$

Kac-Moody Affine Hecke Algebras

Now let G be a Kac-Moody group of (untwisted) affine type, arising from a generalised Cartan matrix, with Weyl group W_{aff}.

The Cartan decomposition only holds for a semigroup $G^+ \subset G(L)$.

The **Kac-Moody affine Hecke algebra** $\widehat{\mathcal{H}}$ is the convolution algebra on $\mathbb{C}$-valued functions of $I \backslash G^+ / I$, defined by BKP for G untwisted affine, and by BPGR in the general case.

They showed that there is a basis indexed by the **double affine Weyl semigroup** $W_{\mathcal{T}} = W_{\text{aff}} \ltimes \mathcal{T}$.

$\mathcal{T}$ is the (integral) Tits cone, defined as the W_{aff}-translates of the dominant coweights of G.

For $x, y \in W_{\mathcal{T}}$, we can write the product in $\widehat{\mathcal{H}}$ as:

$$T_x T_y = \sum_{z \in W_{\mathcal{T}}} c_{x,y}^z T_z, \quad c_{x,y}^z \in \mathbb{Z}[q]$$

BPGR, and independently Muthiah, proved a conjecture by BKP that the **structure constants** $c_{x,y}^z$ are integer-coefficient polynomials in q.

Conjecture (Muthiah, Puskás)

There is exactly one coefficient $c_{x,y}^z \in \mathbb{Z}[q]$ in the product $T_x T_y$ which is non-zero modulo q. In particular, $T_x T_y \equiv \pm T_z \pmod{q}$ for some $z \in W_{\mathcal{T}}$.

Quantum Bruhat Graphs

Schremmer explored a new way to calculate Demazure products in W_{aff}, which relies on the (finite) **quantum Bruhat graph**.

This has a more natural extension to $W_{\mathcal{T}}$, making use of *affine* quantum Bruhat graphs instead.

We take a generalisation of this new formula as the *definition* of a double affine Demazure product, and show it is well-defined and satisfies expected properties.

The **quantum Bruhat graph** $QBG(W)$ for a Weyl group W is a weighted, directed graph with vertex set W and weights in $\Phi^\vee \cup \{0\}$, the set of coroots with 0.

Let $\alpha \in \Phi^+$. There is an edge $w \to ws_\alpha$ if either:

(i) $\ell(ws_\alpha) = \ell(w) + 1$, weight 0, *Bruhat*.

(ii) $\ell(ws_\alpha) = \ell(w) + 1 - 2\mathrm{ht}(\alpha^\vee)$, weight $\alpha^\vee$, *Quantum*.

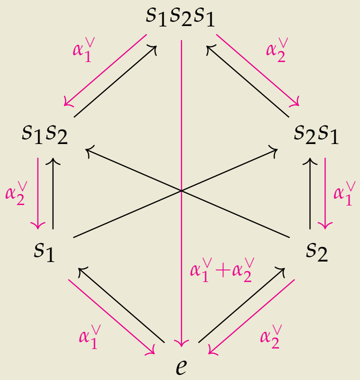
Remark

$QBG(W)$ *is strongly connected, and any two shortest paths have the same total weight.*

Denote $d(u \Rightarrow v)$ for the length of a shortest path, $\mathrm{wt}(u \Rightarrow v)$ for the weight of a shortest path.

Example

Let W be of type SL_3. Then $W = S_3$, generated by $s_1,\ s_2$.



The roots are $\Phi = \pm\{\alpha_1,\ \alpha_2,\ \alpha_1+\alpha_2\}$.

Schremmer's Formula

Let $x = w_x\tau^{\lambda_x},\ y = w_y\tau^{\lambda_y} \in W_{\text{aff}} \cong W \ltimes Q^\vee$.

Choose $(u,v) \in \mathrm{LP}(x) \times \mathrm{LP}(y)$ such that the distance $d(u \Rightarrow w_y v)$ in the (finite) quantum Bruhat graph is minimal amongst all such pairs.

Theorem (Schremmer)

The Demazure product $x * y$ is given by:

$$x * y = w_x u v^{-1} \tau^{vu^{-1}\lambda_x + \lambda_y - v\mathrm{wt}(u \Rightarrow w_y v)}$$

Note that for this to hold, uv^{-1} and $v\mathrm{wt}(u \Rightarrow w_y v)$ must be independent of choice.

Theorem (Schremmer)

The length of the Demazure product is given by:

$$\ell(x * y) = \ell(x) + \ell(y) - d(u \Rightarrow w_y v)$$

Schremmer's formula requires the restrictions $u \in \mathrm{LP}(x), v \in \mathrm{LP}(y)$.

Let $\langle \cdot, \cdot \rangle$ be the pairing between the coweight and root lattices. Let $\alpha \in \Phi$, and $x = w\tau^\lambda \in W_{\mathrm{aff}}$.

(i) The **length functional** is

$$\ell(x, \alpha) = \langle \lambda, \alpha \rangle + \Phi^+(\alpha) - \Phi^+(w\alpha).$$

(ii) The set of **length positive** elements for x is

$$\mathrm{LP}(x) = \{u \in W \mid \ell(x, u\alpha) \geq 0 \ \forall \alpha \in \Phi^+\}.$$

(iii) The set of **distance-minimising pairs** is

$$M_{x,y} = \{(u, v) \in \mathrm{LP}(x) \times \mathrm{LP}(y) \mid d(u \Rightarrow w_y v) \text{ is minimal}\}.$$

Remark

Length positive elements always exist.

Double Affine Demazure Products

Both the QBG and length positivity naturally extend to a double affine setting, hence we make the following conjecture.

Conjecture

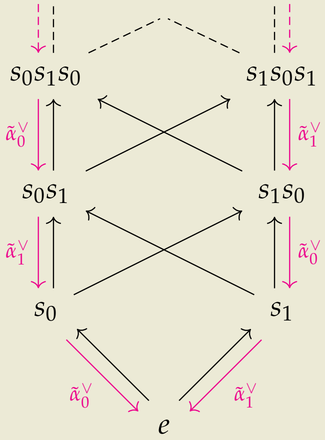
Schremmer's formula extends to a well-defined, associative product on $W_{\mathcal{T}}$, satisfying properties of the Demazure product.

Remark

For $x \in W_{\mathcal{T}}$, the Tits cone condition is necessary for length positive elements to always exist.

Example

Let $W_{aff} = \langle s_0, s_1 \mid s_0^2 = s_1^2 = e \rangle$ *be the Weyl group of type* $\widehat{\mathrm{SL}}_2$.



The roots are $\{\pm\alpha + n\delta \mid n \in \mathbb{Z}\}$, *with* $\tilde{\alpha}_0 = -\alpha + \delta$, $\tilde{\alpha}_1 = \alpha$.

Well-Definedness

We first need to confirm that $x * y$ is well-defined. This requires:

- A distance function on $QBG(W_{\mathrm{aff}})$,
- A weight function,
- uv^{-1} and $v\mathrm{wt}(u \Rightarrow w_y v)$ to be independent of choice of $(u, v) \in M_{x,y}$.

Fortunately, $QBG(W_{\mathrm{aff}})$ is strongly-connected (Welch), and so a distance function is well-defined. However, **it is unknown in general if we have a weight function**.

Our approach is to investigate the case of $\widehat{\mathrm{SL}}_2$, which has a weight function (D.) and a more approachable description of $W_{\mathcal{T}}$.

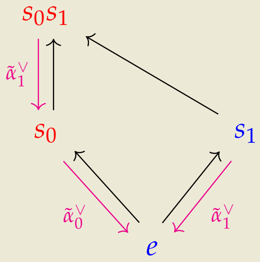
Firstly, we demonstrate that the independence statement is non-trivial, i.e. $|M_{x,y}| > 1$ in many cases.

Example

For $\widehat{\mathrm{SL}}_2$, $\mathcal{T} = \{k\alpha^\vee + m\delta + l\Lambda_0 \mid k, m \in \mathbb{Z}, l \in \mathbb{Z}_{\geq 0}\}$.

Let $x = s_0 s_1 s_0 \varepsilon^{3\Lambda_0}$, $y = s_1 s_0 \varepsilon^{4\alpha^\vee + 2\Lambda_0}$. We want $(u, v) \in \mathrm{LP}(x) \times \mathrm{LP}(y)$ s.t. $d(u \Rightarrow s_1 s_0 v)$ is minimal.

$$\mathrm{LP}(x) = \{e, s_1\}, \ \mathrm{LP}(y) = \{s_0 s_1 s_0, s_0 s_1 s_0 s_1\}, \ s_1 s_0 \mathrm{LP}(y) = \{s_0, s_0 s_1\}.$$



Two possible distance minimising paths:

- $e \Rightarrow s_0$ with weight 0.
- $s_1 \Rightarrow s_0 s_1$ with weight 0.

Hence $M_{x,y} = \{(e, s_0 s_1 s_0), (s_1, s_0 s_1 s_0 s_1)\}$.
In either case, $uv^{-1} = s_0 s_1 s_0$ and $vwt(u \Rightarrow w_y v) = 0$.

Main Results

Theorem (D)

Let $W_{\mathcal{T}}$ be of type $\widehat{\mathrm{SL}}_2$. Then the double affine Demazure product is ***well-defined*** *for $l > 0$, and* ***associative*** *for $l > 1$.*

This is a positive indication that the formula should be well-defined and associative in the general case, for any $W_{\mathcal{T}}$.

Theorem (D)

Let $x, y \in W_{\mathcal{T}}$, and assume that the double affine Demazure product $x * y$ is well-defined. Then

(i) $\ell(x * y) = \ell(x) + \ell(y) \iff x * y = xy$.

(ii) $\ell(x * y) = \ell(x) + \ell(y) - d(u \Rightarrow w_y v)$, for $(u, v) \in M_{x,y}$.

Future Work

- Generalise from $\widehat{\mathrm{SL}}_2$ to all types, and reconnect to the Hecke algebra.
- A well-defined weight function for all affine QBGs.
- Investigate small length deficits, e.g. when $x * y = xsy$.
- Link to a double affine semi-infinite Bruhat order.

Thank You! ☺