Reachability categories and commuting algebras of quivers

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Incidence algebras

Incidence algebras are of interest in combinatorics, algebra, topology.

Recall

The incidence algebra of a poset is *associative*, and it is *finite* if and only if the poset is finite.

In particular, we have connections to:

- posets
- (directed) graphs/quivers
- simplicial complexes.

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The commuting algebra of quivers is the path algebra modulo its parallel ideal.

Theorem (Green - Schroll)

Let Q be a finite quiver and $\mathbb{K}$ a field. Then, the commuting algebra $\mathbb{K}Q/C$ is Morita equivalent to an incidence algebra.

From data to homological invariants

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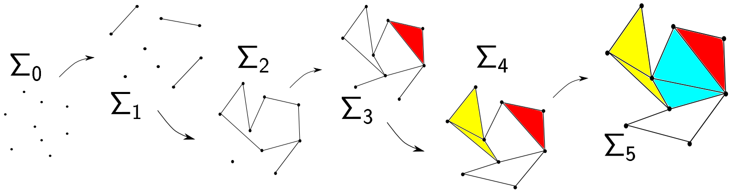

and $f_i^{h,k} \colon \mathrm{H}_i(\Sigma_h; R) \to \mathrm{H}_i(\Sigma_k; R)\ \forall h \leq k$.

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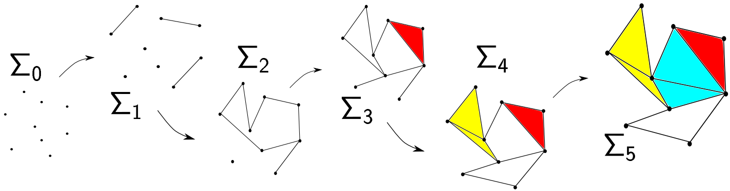
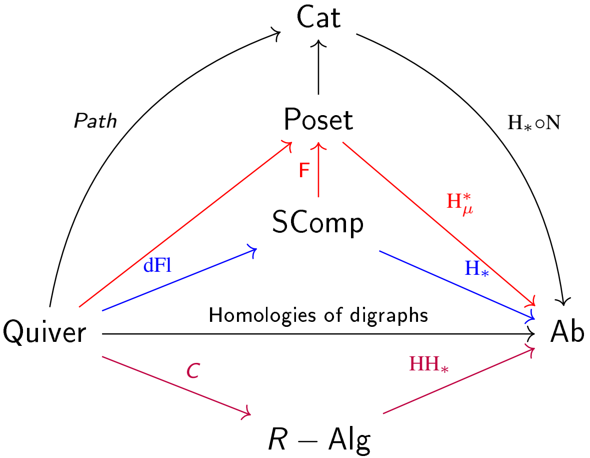


and $f_i^{h,k} \colon \mathrm{H}_i(\Sigma_h; R) \to \mathrm{H}_i(\Sigma_k; R)\ \forall h \leq k$. Persistent homology keeps track of appearance/disappearance of associated homology classes.

(Co)homologies of digraphs

There is a whole zoo of (co)homology theories of directed graphs:



Goal: study functors from graphs to algebras/categories.

Framework

By a quiver we mean a *finite* directed graph $G = (V, E)$. Edges are ordered pairs of vertices. Loops or multiple edges are allowed.

A morphism of quivers from G_1 to G_2 is a function $\varphi\colon V(G_1) \to V(G_2)$ such that:

$$e = (v, w) \in E(G_1) \implies \varphi(e) := (\varphi(v), \varphi(w)) \in E(G_2) \ .$$

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Quiver is the category of finite quivers and morphisms of quivers.

Remark

Let 2 denote the category with objects, and non-identity morphisms $s, t\colon E \to V$. A (finite) quiver is a functor $Q\colon 2 \to \mathrm{Fin}$
$\Rightarrow$ Quiver is its functor category.

(Simple) paths in quivers

A (directed) path from v to w is a sequence $(e_1, \dots, e_n)$ of edges such that $s(e_1) = v$, $t(e_n) = w$, and $t(e_i) = s(e_{i+1})$.
Remark: loops and repetitions are allowed!

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A simple path is a directed path in which no vertex that is encountered more than once.
The length of a simple path is the number of its edges, *e.g.* the length of $(e_0, \dots, e_n)$ is $n + 1$.

Observation

Morphisms of quivers send simple/directed paths to directed paths.

From quivers to categories I

For Q a finite quiver, we consider the small category Path_Q:

- $\mathrm{Ob}(\mathrm{Path}_Q) = V(Q)$;
- for each vertex v there is an identity morphism 1_v corresponding to the trivial path at v;
- morphisms between v and w are all possible *paths* in Q from v to w;
- composition of morphisms is induced by composition of paths.

We call Path_Q the path category of Q.

Question

What is the path category of a single vertex? And of a simple path?

Remark: If Q has directed cycles, then Path_Q is infinite.

From categories to algebras

How to associate algebras to categories?

Definition

Let C be a category and R be a commutative ring with unity. The category algebra $R\mathsf{C}$ is the free R-module with basis the set of morphisms of C.
The product on the basis elements is given by

$$f \cdot g = \begin{cases} f \circ g & \text{when the composition exists in } \mathsf{C} \\ 0 & \text{otherwise} \end{cases}$$

and then it is linearly extended to the whole $R\mathsf{C}$.

Category algebras are associative algebras. If C has finitely many objects, then $R\mathsf{C}$ is also unital. The unit is given by $\sum_{c \in \mathsf{C}} 1_c$.

Examples

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We have several classical examples of category algebras:

- if G is a group, seen as a category, then RG is the classical group algebra;
- if C is the path category Path_Q, then the category algebra $R\mathrm{Path}_Q$ is the classical path algebra of Q;
- Every poset $(P, \leq)$ can be seen as a category P. If P is a finite poset, then the category algebra $R\mathrm{P}$ is isomorphic to the incidence algebra of P.

Remark (Ortega, '06)

Let P be a finite poset. Then, its associated path algebra and incidence algebra are isomorphic if and only if P is a tree (as a poset, i.e. if for each $p \in P$, the set $\{s \in P \mid s < p\}$ is well-ordered).

From quivers to categories II

Let Q be a finite quiver.

Definition

The incidence, or reachability, category Reach_Q is the category with:

$$\mathrm{ob}(\mathrm{Reach}_Q) = V(Q) \ ,$$

and for $v, w \in Q$, has Hom-set

$$\mathrm{Reach}_Q(v, w) := \begin{cases} * & \text{if there is a path from } v \text{ to } w \text{ in } Q \\ \emptyset & \text{otherwise} \end{cases}$$

The Hom-set $\mathrm{Reach}_Q(v, v)$ is defined as the identity on v, given by the trivial path at v.

What is the category algebra of the reachability category?

First properties

A category C is thin if for any pair of objects $c, c' \in C$ there is at most one morphism $c \to c'$ between them.

Proposition

Reachability yields a functor

$$\mathrm{Reach}\colon \mathrm{Quiver} \to \mathrm{Thin}.$$

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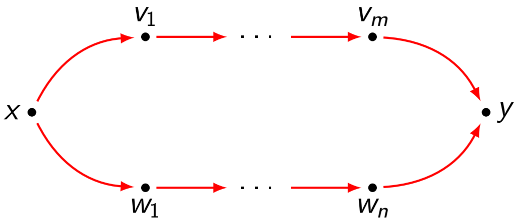
Q is called a polytree if its underlying undirected graph is a tree.

Remark

If Q is a polytree, then $\mathrm{Path}_Q \cong \mathrm{Reach}_Q$*: for all v, w, both* $\mathrm{Reach}_Q(v, w)$ *and* $\mathrm{Path}_Q(v, w)$ *contain at most one morphism.*

Quasi-bigons

Let $B_{m,n}$ be the quiver in figure.
$B_{0,0}$ denotes the quiver on vertices x and y, with two edges from x to y and no other intermediate vertex.



Definition

We say that B is a quasi-bigon of a quiver Q if it is a subquiver of Q isomorphic to $B_{m,n}$ for some $m, n \geq 0$.

A categorical isomorphism

Q is connected if its underlying graph is.

Proposition

Let Q be a finite connected quiver. Then, Path_Q and Reach_Q are isomorphic categories if and only if Q does not contain directed cycles nor quasi-bigons.

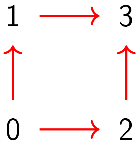
Example: consider the quiver

$$Q = \begin{array}{ccc} 1 & \longleftarrow & 3 \\ \uparrow & & \downarrow \\ 0 & \longrightarrow & 2 \end{array}$$

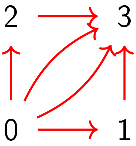
then path category and reachability category of Q are isomorphic.

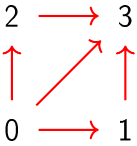
One more example: $B_{1,1}$

Consider the quiver

$Q =$ 

The associated path category and reachability category are not isomorphic. The graph representations of these two categories are:

$\mathrm{Path}_Q =$ 

$\mathrm{Reach}_Q =$ 

where we have omitted the identity morphisms on the vertices.

Towards a topological classification

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- if C has an initial/terminal object, then $|\mathrm{Nerve}(C)|$ is contractible;
- an equivalence of categories $C \cong D$ induces an homotopy equivalence $|\mathrm{Nerve}(C)| \simeq |\mathrm{Nerve}(D)|$ between the respective nerves;

Towards a topological classification

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Proposition (Citterio, '01)

The geometric realization $|\mathrm{Nerve}(\mathrm{Path}_Q)|$ of the nerve of the path category of a quiver Q has the homotopy type of the geometric realisation $|Q|$.

$\Rightarrow \mathrm{Nerve}(\mathrm{Path}_Q)$ is homotopic to a wedge of circles.
For $Q = B_{1,1}$ the square quiver, $\mathrm{Path}_Q \simeq S^1$, $\mathrm{Reach}_Q \simeq *$.

Strongly connected components

A quiver Q is strongly connected if it contains a path from x to y and one from y to x, for every pair of vertices x and y.

Proposition

Let Q be a strongly connected quiver. Then, Reach_Q *is contractible.*

Proof.

- Choose an object q of Q;
- let 1 be the category with one object and a single identity morphism – hence a contractible category;
- write a functor $F\colon 1 \to \mathrm{Reach}_Q$ sending 1 to q;
- F is an equivalence, hence $|\mathrm{Nerve}(\mathrm{Reach}_Q)| \simeq |\mathrm{Nerve}(1)| \simeq *$.

□

A topological equivalence

The condensation $c(Q)$ is the digraph with the strongly connected components of Q as its vertices. There is an edge (X, Y) if there is an edge (x, y) in Q for some $x \in X$ and $y \in Y$.

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Proposition

Let Q be a finite connected quiver with no quasi-bigons. Then, there is a homotopy equivalence

$$|\mathrm{Nerve}(\mathsf{Path}_{c(Q)})| \simeq |\mathrm{Nerve}(\mathsf{Reach}_{Q})| \ .$$

1. Condensation induces the equivalence $|\mathrm{Nerve}(\mathsf{Path}_{c(Q)})| \simeq |\mathrm{Nerve}(\mathsf{Reach}_{c(Q)})|$.
2. Collapsing strongly connected components does not change the homotopy type, hence $|\mathrm{Nerve}(\mathsf{Reach}_{Q})| \simeq |\mathrm{Nerve}(\mathsf{Reach}_{c(Q)})|$.

Quasi-bigons destroy the symmetry

Remark

Let X be a finite simplicial complex and P its face poset.

- $\mathrm{Reach}_P \cong P$;
- *the nerve of* $\mathrm{Reach}_P \cong P$ *is its order complex*
  $\Rightarrow$ *get the barycentric subdivision of* X.

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Consequence: if X has non-trivial homotopy groups in degree ≥ 2, and Q is the Hasse diagram of P, then

$$|\mathrm{Nerve}(\mathrm{Reach}_Q)| \simeq X.$$

But, the homotopy groups of the nerve of a path category are always trivial in dimension ≥ 2.
However, this opens interesting connection to incidence algebras!

Skeletal categories

- A category C is skeletal if each of its isomorphism classes has just one object.
- The skeleton $\mathrm{sk}\, C$ of C is the unique (up to isomorphism) skeletal category equivalent to C.

Remark

Assuming the axiom of choice, every category has a skeleton: choose one object in each isomorphism class of C, and then define $\mathrm{sk}\, C$ to be the full subcategory on this collection of objects.

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Fact: Taking the skeleton induces an equivalence of categories

$$\mathrm{sk}\, C \hookrightarrow C$$

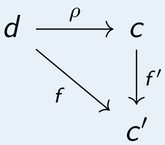
However, this construction can not be promoted to an endofunctor $\mathrm{sk} : \mathrm{Cat} \to \mathrm{Cat}$ of the category of small categories.

Reflections...

Let C be a subcategory of D, and $d \in D$.

Definition

A reflection for d is a morphism $\rho \colon d \to c$ in D from d to $c \in C$ such that the following universal property is satisfied:
for any $f : d \to c'$ in D with $c' \in C$, there exists a unique morphism $f' \colon c \to c'$ of C such that this diagram



commutes.

A subcategory C of D with the property that each object $d \in D$ has a reflection is called a reflective subcategory.

...and associated functors

Let C be a reflective subcategory of D.

Theorem

For each $d \in D$ let $\rho_d \colon d \to c_d$ be a reflection. Then, there exists a unique functor

$$R \colon D \to C$$

such that:

- $R(d) = c_d$ *for all* $d \in D$*;*
- *for each* $f : d \to d'$ *in* D*, the diagram*

$$\begin{array}{ccc} d & \xrightarrow{\rho_d} & R(d) \\ \downarrow f & & \downarrow R(f) \\ d' & \xrightarrow{\rho_{d'}} & R(d') \end{array}$$

commutes

The posetal reflection

Lemma (Borceux - Campanini - Gran - Tholen, '23)

The category of skeletal categories is reflective in Cat.

For P a thin category, set

$$p \simeq q \text{ iff } p \leq q \text{ and } q \leq p \ .$$

$\Rightarrow \rho\colon P \to P/\simeq$, with $P/\simeq$ its reflection. We get a functor

$$L\colon \mathrm{Thin} \to \mathrm{Poset}$$

called the posetal reflection.

Remark

Poset *is a reflective subcategory of* Thin.

The reachability poset

We have constructed a functor

$$\mathcal{R} := L \circ \mathrm{Reach} \colon \mathsf{Quiver} \to \mathsf{Poset}$$

that associates to a quiver the poset resulting from the composition of the reachability functor with the posetal reflection.

Definition

For a finite quiver Q, the poset $\mathcal{R}(Q)$ is called the incidence, or reachability, poset of Q.

Objects of $\mathcal{R}(Q)$ are strongly connected components.
$[v] \leq [w]$ iff there are v and w with a path from v to w.

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Remark

Reach_Q and $\mathcal{R}(Q)$ are equivalent categories!

Application I: commuting algebras

For a finite quiver Q and $\mathbb{K}$ a field, let $\mathbb{K}Q/C$ be the commuting algebra of Q: the path algebra $\mathbb{K}Q$ of Q modulo its parallel ideal C.

Lemma

The category algebra of Reach_Q *is isomorphic to the commuting algebra* $\mathbb{K}Q/C$.

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Idea: If Q is a finite quiver, and $\mathbb{K}$ a field, then the category algebra of Path_Q is the classical path algebra. Consider the map

$$\mathbb{K}\mathrm{Path}_Q \longrightarrow \mathbb{K}\mathrm{Reach}_Q$$

of vector spaces, induced by $F\colon \mathrm{Path}_Q \to \mathrm{Reach}_Q$ (which collapses all paths between pairs of objects to a single morphism). The kernel is the parallel ideal of $\mathbb{K}Q$ and composition of paths is preserved.

Morita equivalence

Definition

Two unital rings are said to be Morita equivalent if and only if their categories of left (or right) modules are equivalent.

What about category algebras?

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Proposition

*If C and D are equivalent categories with **finitely** many objects, and R is a (unital) commutative ring, then the category algebras RC and RD are Morita equivalent.*

$\Rightarrow$ if Q is a finite quiver and R a unital commutative ring, the category algebras of Reach_Q and $\mathcal{R}(Q)$ are Morita equivalent.

Commuting algebras are incidence algebras

Theorem (Green - Schroll)

Let Q be a finite quiver and $\mathbb{K}$ a field. Then, the commuting algebra $\mathbb{K}Q/C$ is Morita equivalent to the incidence algebra of $\mathcal{R}(Q)$.

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- The category algebra of Reach_Q is isomorphic to the commuting algebra of Q.

$\Rightarrow \mathbb{K}Q/C$ is Morita equivalent to the category algebra of the reachability poset $\mathcal{R}(Q)$, which is an incidence algebra.

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- Q finite $\Rightarrow$ the category algebras are Morita equivalent.
- The category algebra of Reach_Q is isomorphic to the commuting algebra of Q.

$\Rightarrow$ $\mathbb{K}Q/C$ is Morita equivalent to the category algebra of the reachability poset $\mathcal{R}(Q)$, which is an incidence algebra. Moreover:

Theorem

The commuting algebras of Q, Q' are Morita equivalent if and only if $\mathcal{R}(Q) \cong \mathcal{R}(Q')$.

A converse result

If Q does not contain quasi-bigons, then the path algebra of its condensation is Morita equivalent to an incident algebra.

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Remark

It is a result of Stanley that going from (locally) finite posets to incidence algebras is conservative.

I.e. if the incidence algebras of two locally finite posets P and Q are isomorphic, as $\mathbb{K}$-algebras, then also P and Q are isomorphic.

Corollary

Let $\mathbb{K}$ be a field. If the commuting algebras of the finite posets P and Q are isomorphic, as $\mathbb{K}$-algebras, then the reachability categories Reach_P and Reach_Q are isomorphic.

A bound on the global dimension

- The global dimension of a ring R is the supremum of the set of projective dimensions of all R-modules.
- Let $R(Q)$ be the underlying quiver of $\mathcal{R}(Q)$ and $\mathrm{diam}(Q)$ the maximal length across all directed *simple* paths in $R(Q)$.

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Corollary

Let Q be a finite quiver. Then, $\mathrm{gl.dim}\,\mathbb{K}Q/C \leq \mathrm{diam}(Q)$.

Idea: if $\ell(\mathrm{Reach}_Q)$ is the maximal length of chains of non-isomorphisms in $\mathcal{R}(Q)$, we have

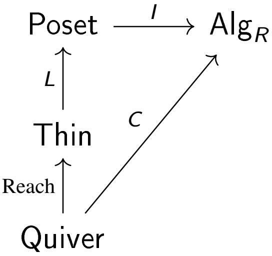
$$\mathrm{gl.dim}\,\mathbb{K}\mathrm{Reach}_Q \leq \ell(\mathrm{Reach}_Q) = \mathrm{diam}(Q) \ .$$

Corollary

Let Q be a finite quiver with at least one edge. Then, $\mathrm{gl.dim}\,\mathbb{K}Q/C \leq 1$. iff any closed interval of $\mathcal{R}(Q)$ is totally ordered.

Possible enhancement?

We can depict the story on a diagram:



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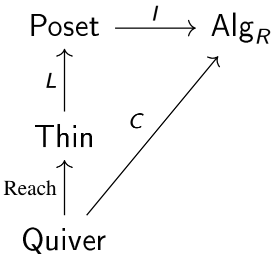

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Remark (Dell'Ambrogio - Tabuada, '12)

There is a Quillen model structure on Cat_R *for which Morita equivalences are homotopy equivalences.*

Is there a homotopical enhancement of the result in Quiver?

Hochschild cohomology

Homological invariants for Path_Q vanish beyond degree 1.

Theorem (Happel)

If Q is a connected quiver without oriented cycles and $\mathbb{K}$ is an algebraically closed field, then

$$\dim_{\mathbb{K}} \mathrm{HH}^i(A) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{if } i > 1 \\ 1 - n + \sum_{e \in E} \dim_{\mathbb{K}} e_{t(e)} A e_{s(e)} & \text{if } i = 1 \end{cases}$$

where $A = \mathbb{K}\mathrm{Path}_Q$, $n = |V(Q)|$ and $e_{t(e)} A e_{s(e)}$ is the subspace of A generated by all the possible paths from $s(e)$ to $t(e)$ in Q.

However, $\mathrm{HH}^*(I(F(X))) \cong H^*(X)$ and simplicial cohomology is Hochschild cohomology.

Back to TDA

By Morita equivalence, we have

$$\beta_*^{\mathrm{HH}}(\mathbb{K}\mathrm{Reach}_Q) = \beta_*^{\mathrm{HH}}(\mathbb{K}\mathcal{R}(Q)) \ .$$

Then, the composition

$$(\mathbb{R}, \leq) \to \mathrm{Quiver} \xrightarrow{L\circ\mathrm{Reach}} \mathrm{Poset} \xrightarrow{\mathbb{K}\text{-}} \mathbb{K}\text{-}\mathrm{Alg} \xrightarrow{\beta_*^{\mathrm{HH}}} \mathbb{N} \ .$$

are the Betti curves of the associated *persistent Hochschild (co)homology groups*, aka persistent HH-curves.

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$$\beta_*^{\mathrm{HH}}(\mathbb{K}\mathrm{Reach}_Q) = \beta_*^{\mathrm{HH}}(\mathbb{K}\mathcal{R}(Q)) \ .$$

Then, the composition

$$(\mathbb{R}, \leq) \to \mathrm{Quiver} \xrightarrow{L \circ \mathrm{Reach}} \mathrm{Poset} \xrightarrow{\mathbb{K}\text{-}} \mathbb{K}\text{-}\mathrm{Alg} \xrightarrow{\beta_*^{\mathrm{HH}}} \mathbb{N} \ .$$

are the Betti curves of the associated *persistent Hochschild (co)homology groups*, aka persistent HH-curves.

Theorem

Let $F\colon (\mathbb{R}, \leq) \to \mathrm{Quiver}$ *be a filtration of quivers. Then, the persistent HH-curves agree with the simplicial Betti curves of the nerves of the reachability categories.*

The MPSS

Magnitude generalizes the notion of Euler characteristic to arbitrary categories and was categorified by Hepworth-Willerton.
Consider the spectral sequences associated to the length filtration $F_* N(\mathrm{Reach}_Q) \subseteq N(\mathrm{Reach}_Q)$; this is called the magnitude-path spectral sequence.

Theorem (Asao)

The magnitude-path spectral sequence $IE^{*,*}_*$ satisfies the following:

1. *The first page coincides with magnitude homology of Q.*
2. *The diagonal of the second page $IE^{*,2}_*$ is the path homology of Q.*
3. *It converges to the homology of the nerve of the reachability category $N(\mathrm{Reach}_Q)$.*

Injective words on quivers

Recall: The complex of injective words $\Delta(W)$ on a set W is the simplicial complex on all ordered sequences of distinct elements of W.

Theorem (Farmer, '79)

For $|W| = n$, the complex $\Delta(W)$ is homotopy equivalent to a wedge of $(n-1)$-spheres.

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Recall: The complex of injective words $\Delta(W)$ on a set W is the simplicial complex on all ordered sequences of distinct elements of W.

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For a quiver Q, we define the complex of injective words $\Delta(Q)$ on Q as the complex of words on $V(Q)$ with the arrows constrains.

Theorem (C.-Menara, '25)

The complex $\Delta(Q)$ is homotopic to the injective nerve $N^{\iota}\mathrm{Reach}_Q$.

$\Rightarrow$ if Q is a poset, then $\Delta(Q)$ is homotopic to $N\mathrm{Reach}_Q$
$\Rightarrow$ $\Delta(Q)$ is generally not a wedge of spheres.

Conclusions and questions

- We have analysed path and reachability categories. What can we say about categories "*in between*"? what can we say about other quotient ideals? or quasi-commuting algebras?
- Invariants associated to path categories are *1-dimensional*, whereas invariants associated to reachability categories are not, and we have shown some bounds. Can we extend these results to more general rings?
- When quivers are finite, the associated categories are finite, and the algebras unital. How much can we say about infinite quivers?
- Morita equivalence is an equivalence in a suitable Quillen model category. Can we use it to describe a model category on graphs which reflects this part of the story?