Extension between simple and costandard $(\mathfrak{g}, B)$-modules

(joint with Simon Riche)

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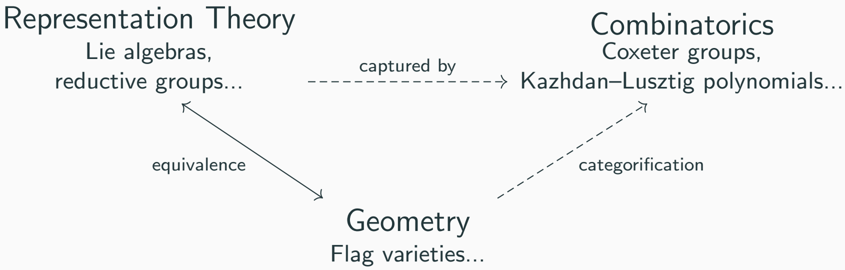
Collaborations in Algebra, Representation Theory and Ethics


Representation Theory
Lie algebras,
reductive groups...
captured by
Combinatorics
Coxeter groups,
Kazhdan–Lusztig polynomials...
equivalence
categorification
Geometry
Flag varieties...

BGG category $\mathcal{O}$

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$\mathfrak{g}$ complex semisimple Lie algebra

triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{t} \oplus \mathfrak{n}$, with Borel $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}$

Definition (Bernstein–Gelfand–Gelfand)

The category $\mathcal{O}$ for $\mathfrak{g}$ consists of finitely generated $\mathfrak{g}$-modules on which

- $\mathfrak{t}$-action is semisimple;
- $\mathfrak{n}$-action is locally nilpotent.

For example, for each $\lambda \in \mathfrak{t}^*$ we have Verma module

$$\Delta(\lambda) = \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{b}} \mathbb{C}_\lambda.$$

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Standard, costandard and simple

Let $\lambda \in \mathfrak{t}^*$.

- standard module $\Delta(\lambda) = \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{b}} \mathbb{C}_\lambda$, which satisfies an isomorphism

$$\mathrm{Hom}_{\mathfrak{g}}(\Delta(\lambda), -) \simeq \mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda, -).$$

- simple module $L(\lambda)$, which appears as the unique simple quotient of $\Delta(\lambda)$.
- costandard module $\nabla(\lambda)$, which is uniquely determined by the functorial isomorphism

$$\mathrm{Hom}_{\mathfrak{g}}(-, \nabla(\lambda)) \simeq \mathrm{Hom}_{\mathfrak{b}^-}(-, \mathbb{C}_\lambda).$$

$L(\lambda)$ appears as the unique simple submodule in $\nabla(\lambda)$.

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Character of simple modules

The character of $M \in \mathcal{O}$ is the formal sum

$$\mathrm{ch} M := \sum_{\lambda \in \mathfrak{t}^*} \dim M_\lambda \cdot e^\lambda.$$

For example, since $\Delta(\lambda) = \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{b}} \mathbb{C}_\lambda \simeq \mathcal{U}\mathfrak{n}^- \otimes \mathbb{C}_\lambda$, we have

$$\mathrm{ch}\Delta(\lambda) = e^\lambda \prod_{\alpha>0} \frac{1}{1-e^{-\alpha}} \overset{\text{Fact}}{=\!=} \mathrm{ch}\nabla(\lambda).$$

Problem (Fundamental, but hard!)

Compute $\mathrm{ch} L(\lambda)$ *?*

For example, if λ is dominant, then we have Weyl's character formula

$$\mathrm{ch} L(\lambda) = \frac{\sum_{w\in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)}}{\prod_{\alpha>0} (e^{\alpha/2} - e^{-\alpha/2})},$$

where W is the Weyl group and ρ is halfsum of positive roots.

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Highest weight category

Character map ch factors through $K(\mathcal{O})$. To know $\mathrm{ch}L(\lambda)$, it is equivalent to know

$$[L(\lambda)] = \sum_{\mu \in \mathfrak{t}^*} ? \cdot [\Delta(\mu)].$$

Fact (Highest weight category)

The category $\mathcal{O}$ admits a highest weight structure, with standard objects $\{\Delta(\lambda)\}_{\lambda \in \mathfrak{t}^*}$ and costandard objects $\{\nabla(\lambda)\}_{\lambda \in \mathfrak{t}^*}$. In particular,

$$\mathrm{Ext}^i_{\mathcal{O}}(\Delta(\lambda), \nabla(\mu)) = \delta_{i,0}\delta_{\lambda,\mu}\mathbb{C}.$$

$\Rightarrow$ For any $M \in \mathcal{O}$,

$$[M] = \sum_{\mu \in \mathfrak{t}^*} \sum_i (-1)^i \dim \mathrm{Ext}^i_{\mathcal{O}}(M, \nabla(\mu)) \cdot [\Delta(\mu)].$$

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Principal block

$\bullet$-action of W on $\mathfrak{t}^*$ is by

$$w \bullet \lambda = w(\lambda + \rho) - \rho, \quad \forall \lambda \in \mathfrak{t}^*.$$

Proposition (Block decomposition)

There is a decomposition

$$\mathcal{O} = \bigoplus_{\omega \in \mathfrak{t}^*/(W,\bullet)} \mathcal{O}^\omega$$

s.t. $\Delta(\lambda)$ (equivalently, $L(\lambda)$) is contained in $\mathcal{O}^\omega$ iff $W \bullet \lambda = \omega$.

To study $\mathcal{O}$, it is "enough" to study the principal block $\mathcal{O}^0$, in which we have

$$\Delta_w := \Delta(w^{-1} \bullet (-2\rho)), \quad \nabla_w := \nabla(w^{-1} \bullet (-2\rho)), \quad L_w := L(w^{-1} \bullet (-2\rho)).$$

Conjecture (Kazhdan–Lusztig)

$\sum_i \dim \mathrm{Ext}^i_{\mathcal{O}}(L_w, \nabla_y) \cdot v^i =$ Kazhdan–Lusztig polynomial.

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Hecke algebra

The Hecke algebra $\mathcal{H}(W) = \bigoplus_{w \in W} \mathbb{Z}[v^{\pm 1}] \cdot H_w$ is a $\mathbb{Z}[v^{\pm 1}]$-algebra by (for any $w \in W$ and simple reflection s)

$$H_w \cdot H_s = \begin{cases} H_{ws} & \text{if } ws > w \\ H_{ws} - (v - v^{-1}) H_w & \text{if } ws < w. \end{cases}$$

There is an algebra automorphism $\overline{?}$ on $\mathcal{H}(W)$ s.t. $\overline{v} = v^{-1}$, $\overline{H_w} = H_{w^{-1}}^{-1}$.

Proposition (Kazhdan–Lusztig)

There exists unique $\mathbb{Z}[v^{\pm 1}]$-basis $\{C_w\}_{w \in W}$ (canonical basis) s.t.

- $\overline{C_w} = C_w$;
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The Kazhdan–Lusztig polynomials $\{h_{y,w}(v)\}_{y,w \in W}$ are

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Flag variety

Let G be a semisimple algebraic group with $\mathrm{Lie}(G) = \mathfrak{g}$, and B be Borel subgroup with $\mathrm{Lie}(B) = \mathfrak{b}$.

The flag variety G/B admits a stratification into B-orbits

$$G/B = \bigsqcup_{w \in W} BwB/B.$$

Denote by $i_w : BwB/B \hookrightarrow G/B$ be the embedding.

The Schubert variety is $\mathfrak{S}_w = \overline{BwB/B}$, which is usually singular.

Let $\mathrm{IC}_{\mathfrak{S}_w}$ be the IC complex of $\mathfrak{S}_w$.

Theorem (Kazhdan–Lusztig)

$$\sum_i \mathrm{rk}\ {}^{\mathrm{p}}\mathcal{H}^{-i}(\mathrm{IC}_{\mathfrak{S}_w}|_{ByB/B}) \cdot v^i = h_{y,w}(v).$$

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Equivalence

Theorem (Beilinson–Bernstein, Kashiwara–Brylinski)

There is an equivalence of categories

$$\mathcal{O}^0 \;\simeq\; \mathrm{Perv}_{(B)}(G/B),$$

$$L_w \mapsto \mathrm{IC}_{\mathfrak{S}_w}, \quad \Delta_w \mapsto (i_w)_! \mathbb{C}_{BwB/B}[\ell(w)], \quad \nabla_w \mapsto (i_w)_* \mathbb{C}_{BwB/B}[\ell(w)].$$

$\Rightarrow$

$$\dim \mathrm{Ext}^i_{\mathcal{O}}(L_w, \nabla_y) = \mathrm{rk}\ {}^{\mathrm{p}}\mathcal{H}^{-i}(\mathrm{IC}_{\mathfrak{S}_w}|_{ByB/B}),$$

which thus implies Kazhdan–Lusztig's conjecture.

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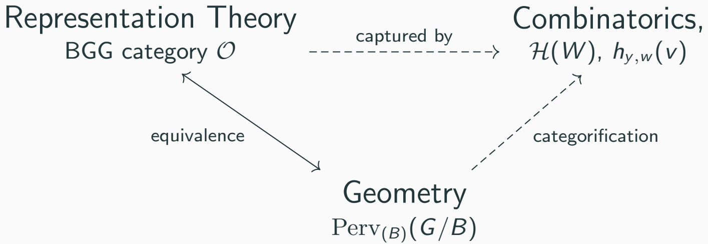
$$L_w \mapsto \mathrm{IC}_{\mathfrak{S}_w}, \quad \Delta_w \mapsto (i_w)_! \mathbb{C}_{BwB/B}[\ell(w)], \quad \nabla_w \mapsto (i_w)_* \mathbb{C}_{BwB/B}[\ell(w)].$$

$\Rightarrow$

$$\dim \mathrm{Ext}^i_{\mathcal{O}}(L_w, \nabla_y) = \mathrm{rk}\ {}^{\mathrm{p}}\mathcal{H}^{-i}(\mathrm{IC}_{\mathfrak{S}_w}|_{ByB/B}),$$

which thus implies Kazhdan–Lusztig's conjecture.

Geometric Representation of BGG category $\mathcal{O}$



$(\mathfrak{g}, B)$-modules

Modular BGG category $\mathcal{O}$

G reductive group $\supset B$ Borel $\supset T$ Cartan, defined over $\Bbbk = \overline{\Bbbk}$ with $\mathrm{char}(\Bbbk) = p > 0$

triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{t} \oplus \mathfrak{n}$

Definition

1. $(\mathfrak{g}, B)$-modules are B-equivariant $\mathfrak{g}$-modules, on which the differential of B-action and the $\mathfrak{g}$-action coincide on $\mathfrak{b}$.
2. The modular BGG category $\mathcal{O}$ is defined as

$$(\mathfrak{g}, B)-\mathrm{Mod}.$$

If $\Bbbk$ were $\mathbb{C}$, then $(\mathfrak{g}, B)-\mathrm{Mod}$ is the full subcategory of BGG category $\mathcal{O}$ of modules whose $\mathfrak{t}$-weights are integrable.

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Standard, costandard and simple

Let $\lambda \in X^*(T) = \mathrm{Hom}(T, \Bbbk^\times)$.

- standard module $\Delta(\lambda) = \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{b}} \Bbbk_\lambda$, which satisfies

$$\mathrm{Hom}_{(\mathfrak{g},B)}(\Delta(\lambda), -) \simeq \mathrm{Hom}_B(\Bbbk_\lambda, -).$$

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Principal block

Let $\mathfrak{R} \subset X^*(T)$ be root lattice.

Let $W_{\text{aff}} = W \ltimes \mathfrak{R}$ be the affine Weyl group, and denote $t_\lambda = 1 \ltimes \lambda$. Consider the $\bullet_p$-action of W_{aff} on $X^*(T)$ by

$$(t_\lambda w) \bullet_p \mu = w(\mu + \rho) - \rho + p\lambda, \quad \forall w \in W, \lambda \in \mathfrak{R}, \mu \in X^*(T).$$

Proposition (Block decomposition)

There is a decomposition

$$(\mathfrak{g}, B)\text{–Mod} = \bigoplus_{\omega \in X^*(T)/(W_{\text{aff}}, \bullet_p)} (\mathfrak{g}, B)\text{–Mod}^\omega$$

s.t. $\Delta(\lambda)$ *is contained in* $(\mathfrak{g}, B)\text{–Mod}^\omega$ *iff* $W_{\text{aff}} \bullet_p \lambda = \omega$.

The principal block $(\mathfrak{g}, B)\text{–Mod}^0$ contains

$$\Delta_x := \Delta(x \bullet_p (-2\rho)), \quad \nabla_x := \nabla(x \bullet_p (-2\rho)), \quad L_x := L(x \bullet_p (-2\rho)), \quad x \in W_{\text{aff}}.$$

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Periodic Hecke module

- W_{aff} is a Coxeter group, and thus endowed with Bruhat order $\leq$.
  We define the semi-infinite order $\leq^{\frac{\infty}{2}}$ on W_{aff} by

$$x \leq^{\frac{\infty}{2}} y \quad \text{iff} \quad t_\lambda x \leq t_\lambda y \text{ when } \lambda \text{ is dominant enough.}$$

- The affine Hecke algebra $\mathcal{H}(W_{\mathrm{aff}})$ acts on periodic Hecke module

$$\mathcal{P}(W_{\mathrm{aff}}) = \bigoplus_{x \in W_{\mathrm{aff}}} \mathbb{Z}[v^{\pm 1}] \cdot H_x^{\frac{\infty}{2}}$$

  by (for any $x \in W_{\mathrm{aff}}$ and affine simple reflection s)

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Main result

Theorem (Riche–S.)

Suppose $p = \mathrm{char}(\Bbbk)$ is large enough.

For any $x, y \in W_{\mathrm{aff}}$, we have

$$\sum_i \dim \mathrm{Ext}^i_{(\mathfrak{g},B)}(L_x, \nabla_y) \cdot v^i = p_{y,x}(v).$$

Remark

1. The proof uses a “Koszul duality” relating $(\mathfrak{g}, B)$-modules to G_1T-modules, based on localization theory of $\mathfrak{g}$-modules and linear Koszul duality for coherent sheaves.
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$\check{G}$: Langlands dual group to G, with $\check{B}$, $\check{T}$, $\check{U} = R_u(\check{B})$, defined over $\mathbb{C}$.

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Theorem (Achar–Dhillon–Riche, upcoming)

There is a category equivalence

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- loop group $\check{G}(\mathbb{C}(\!(t)\!))$, arc group $\check{G}(\mathbb{C}[\![t]\!])$ with $\check{G}(\mathbb{C}[\![t]\!]) \xrightarrow{\mathrm{ev}_{t=0}} \check{G}$;
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- (enhanced) affine flag variety $\mathcal{F}l = \check{G}(\mathbb{C}(\!(t)\!))/I_u$;
- semi-infinite Iwahori subgroup $I^{\frac{\infty}{2}} := \check{U}(\mathbb{C}(\!(t)\!))\,\check{T}(\mathbb{C}[\![t]\!])$;
- $i_x : I^{\frac{\infty}{2}} x I_u / I_u \hookrightarrow \mathcal{F}l$ the embedding;
- We have semi-infinite Schubert variety $\mathfrak{S}_x^{\frac{\infty}{2}} = \overline{I^{\frac{\infty}{2}} x I_u / I_u}$.

Theorem (Achar–Dhillon–Riche, upcoming)

There is a category equivalence

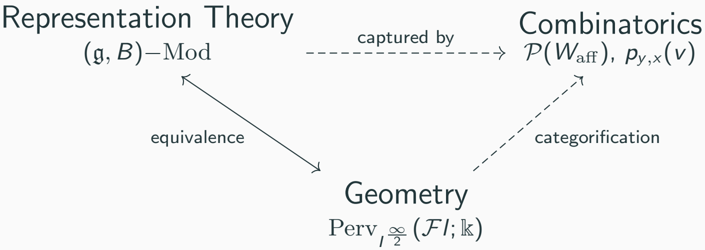
$$(\mathfrak{g}, B){-}\mathrm{Mod}^0 \;\simeq\; \mathrm{Perv}_{I^{\frac{\infty}{2}}}(\mathcal{F}l; \Bbbk),$$

$$L_x \mapsto \mathrm{IC}_{\mathfrak{S}_x^{\frac{\infty}{2}}}, \quad \nabla_x \mapsto (i_x)_* \Bbbk_{I^{\frac{\infty}{2}} x I_u / I_u}[dim].$$

Corollary (conjectured by Achar–Dhillon–Riche)

We have $\sum_i \mathrm{rk}\, {}^{\mathrm{p}}\mathcal{H}^{-i}(\mathrm{IC}_{\mathfrak{S}_x^{\frac{\infty}{2}}}|_{I^{\frac{\infty}{2}} y I_u / I_u}) \cdot v^i = p_{y,x}(v)$.

Geometric Representation of $(\mathfrak{g}, B)-\mathrm{Mod}$



Thank you!