Connections between Lie algebras, toward Cyclotomic KLR algebras

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Framework

Take $n, r \geq 1$ two integers.

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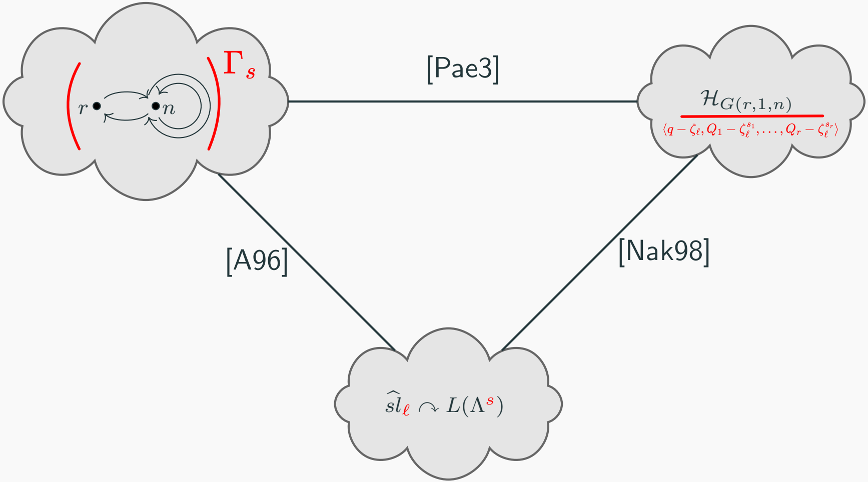
Take $\ell \geq 1$ an other integer, and $s \in (\mathbb{Z}/\ell\mathbb{Z})^r$.



I - Gieseker space

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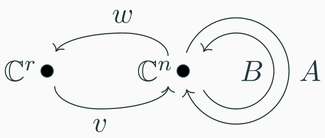
From $n, r \geq 1$.



Denote by

- $\mathrm{Rep}(Q_\bullet, n, r) := \mathrm{M_n}(\mathbb{C})^2 \oplus \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^r)$ is symplectic.
- $\mathrm{GL}_n(\mathbb{C}) \curvearrowright \mathrm{Rep}(Q_\bullet, n, r)$: $g.(A, B, v, w) = (gAg^{-1}, gBg^{-1}, gv, wg^{-1})$.
- Momentum map $\mu_{Q_\bullet} : \begin{array}{ccc} \mathrm{Rep}(Q_\bullet, n, r) & \to & \mathrm{M}_n(\mathbb{C}) \\ (A, B, v, w) & \mapsto & AB - BA + vw \end{array}$.

I - Gieseker space

- The Gieseker space is:

$$\begin{aligned}\mathcal{G}(n,r) &= \mu_{Q_\bullet}^{-1}(0) /\!\!/_{\det} \mathrm{GL}_n(\mathbb{C}) \\ &\simeq \mu_{Q_\bullet}^{-1}(0)^{\det-ss} /\!\!/ \mathrm{GL}_n(\mathbb{C}).\end{aligned}$$

Facts:

- The Gieseker space is a smooth algebraic variety of dimension $2nr$.
- It is equipped with a $\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_r(\mathbb{C})$-action:

$$^{(g,h)}(A, B, v, w) := (aA + bB, cA + dB, vh^{-1}, \det(g)hw),$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

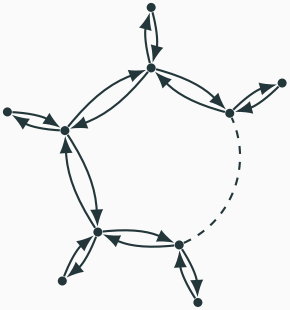
I - Gieseker space

With $\ell \geq 1$ and $s \in (\mathbb{Z}/\ell\mathbb{Z})^r$.

Let $\Gamma_s := \langle \begin{pmatrix} \zeta_\ell & 0 \\ 0 & \zeta_\ell^{-1} \end{pmatrix}, \mathrm{diag}(\zeta_\ell^{s_1}, \ldots, \zeta_\ell^{s_r}) \rangle < \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_r(\mathbb{C})$.

Question: What do irreducible components of $\mathcal{G}(n, r)^{\Gamma_s}$ look like?

Answer: Nakajima quiver varieties over the following quiver:



II - Ariki-Koike algebra

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Let $\mathcal{H}_{n,r}(\ell, s)$:= specialisation of the Hecke algebra of $G(r, 1, n) := (\mathbb{Z}/r\mathbb{Z})^n \rtimes \mathfrak{S}_n$ at ζ_ℓ, $\zeta_\ell^{s_1}, \dots, \zeta_\ell^{s_r}$.

Facts:

- (•) The $\mathbb{C}$-algebra $\mathcal{H}_{n,r}(\ell, s)$ is of dimension $n!r^n$ (and is not semisimple).
- (•) Krull-Schmidt theorem implies that there is a unique (up to permutation) decomposition into indecomposable two-sided ideals:

$$\mathcal{H}_{n,r}(\ell, s) = B_1 \oplus \cdots \oplus B_k.$$

Each B_i is called a block of the Ariki-Koike algebra and

blocks control the representation theory of $\mathcal{H}_{n,r}(\ell, s)$.

III - Algebraic combinatorics

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Notation:

- $\lambda := (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0) \vdash n$ i.e. $|\lambda| := \sum_{i=1}^{k} \lambda_i = n$.
- $\lambda^\bullet := (\lambda^1, \lambda^2, \ldots, \lambda^r) \vdash_r n$ i.e. $|\lambda^\bullet| := \sum_{i=1}^{r} |\lambda^i| = n$. Let $\mathcal{P}_n^r := \{\lambda^\bullet \vdash_r n\}$.
- $\mathrm{Res}_\ell : \mathcal{P}_n^r \times (\mathbb{Z}/\ell\mathbb{Z})^r \to Q$.

III - Algebraic combinatorics

$\mathcal{H}_{n,r}(\ell, s)$ is a cellular algebra [GL96].
$\implies$ Cell modules called Specht modules: $\{S^{\lambda^\bullet} \mid \lambda^\bullet \in \mathcal{P}_n^r\}$.

Remark
These modules are special because:

- All composition factors of $S^{\lambda^\bullet}$ are in the same block,
- simple modules of $\mathcal{H}_{n,r}(\ell, s)$ are given by $S^{\lambda^\bullet}/\mathrm{rad}(S^{\lambda^\bullet}) \neq 0$.

Question: When are $S^{\lambda^\bullet}$ and $S^{\mu^\bullet}$ in the same block of $\mathcal{H}_{n,r}(\ell, s)$?

Answer: Exactly when $\mathrm{Res}_\ell(\lambda^\bullet, s) = \mathrm{Res}_\ell(\mu^\bullet, s)$ [LM07].

Let $\mathcal{T} := \mathbb{T}_2 \times \mathbb{T}_r < \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_r(\mathbb{C})$

$$\mathcal{G}(n,r)^{\mathcal{T}} \overset{1:1}{\longleftrightarrow} \mathcal{P}_n^r. \quad \text{[NakYo]}$$

Question: In which I.C. of $\mathcal{G}(n,r)^{\Gamma_s}$ is the $\mathcal{T}$-fixed point indexed by $\lambda^\bullet \in \mathcal{P}_n^r$?

Answer: In the I.C. indexed by $\mathrm{Res}_\ell(\lambda^\bullet, s)$.

Definition

- ℓ-core of a partition: $\ell = 4$, $\lambda = (3, 3, 3) \rightsquigarrow (1)$.
- ℓ-multicore is a multipartition $\lambda^\bullet$ such that each partition λ^i is an ℓ-core.
- ℓ-core of a charged multipartition $(\lambda^\bullet, s) \rightsquigarrow (\gamma^\bullet, s')$.
- ℓ-core block ([Fay07]) is a block of $\mathcal{H}_{n,r}(\ell, s)$ containing only ℓ-multicores.

Theorem ([Pae3])

Let B be a block of $\mathcal{H}_{n,r}(\ell, s)$.
The dimension parameter d of the I.C. of $\mathcal{G}(n,r)^{\Gamma_s}$ associated with B controls whenever B is an ℓ-core block.
Moreover from d, it is possible to determine the multicharge s' of the ℓ-core of any multipartition in B.

Conclusion

- KLR algebras are graded algebras that were independently introduced by Khovanov–Lauda and Rouquier.
- An important feature is that they categorify the positive part of quantised Kac-Moody algebras.
- Cyclotomic KLR algebras are finite dimensional quotients of KLR algebras.
- In [BK09], it is proven that blocks of Ariki-Koike algebras are isomorphic to blocks of cyclotomic KLR algebras in type A. Cyclotomic KLR algebras can be defined for quivers of type D and E.

Question: Do the connections between fixed points in the Gieseker space and blocks of cyclotomic KLR algebras extend to type D and E?

Thank you for your attention !

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