Finiteness Aspects in Monoids of Endomorphisms of Projective Varieties

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Collaborations in Algebra, Representation Theory and Ethics

Plan

Motivation and Preliminary Notions

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Let X be a projective algebraic variety over an algebraically closed field k (e.g. the field $\mathbb{C}$ of complex numbers).

$\mathrm{End}(X)$ is a monoid under composition.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend left, "g \circ f"] \arrow[rrr, bend left=50, "h \circ g \circ f"] & X \arrow[r, "g"] \arrow[rr, bend right, "h \circ g"'] & X \arrow[r, "h"] & X
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}"] \arrow[rr, bend left, "f \circ \mathrm{id} = f"] & X \arrow[r, "f"] \arrow[rr, bend right, "\mathrm{id} \circ f = f"'] & X \arrow[r, "\mathrm{id}"] & X
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What about geometry ?

There exists a "universal object," namely a scheme, denoted by End_X, parametrizing all endomorphisms of X, and such that $\mathrm{End}(X)$ is the set of points of End_X over k.

Endomorphism Scheme

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Connected Components

The set of connected components of End_X inherits its monoid structure; we denote this monoid by $\pi_0 \mathrm{End}_X$, it is at most countable.

We have a canonical homomorphism of monoids

$$q : \mathrm{End}_X \to \pi_0 \mathrm{End}_X \, .$$

Similarly, we have

- $\mathrm{Aut}(X)$: the group of automorphisms of X,
- Aut_X: the open subgroup of units of End_X, and
- $q' : \mathrm{Aut}_X \to \pi_0 \mathrm{Aut}_X$: a homomorphism of groups to the group of connected components of Aut_X, a restriction of q.

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Automorphisms

Theorem

$f \in \mathrm{End}(X)$ is an automorphism if and only if f is of degree one. In other words,

$$\mathrm{Bir}(X) \cap \mathrm{End}(X) = \mathrm{Aut}(X),$$

where $\mathrm{Bir}(X) =$ the group of birational transformations of X.

Proposition (known)

The degree is constant on each connected component of End_X.

Corollary

- Aut_X is a union of connected components of End_X.
- The group $\pi_0 \mathrm{Aut}_X$ is the subgroup of invertible elements of the monoid $\pi_0 \mathrm{End}_X$.

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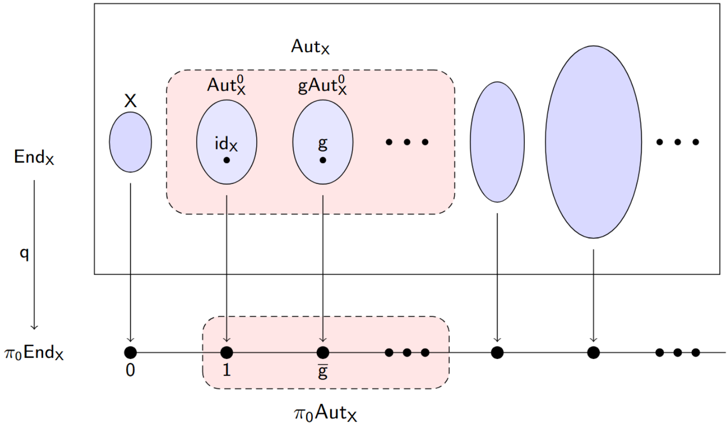
Map of End_X



Example: smooth projective curves

We distinguish three cases depending on the genus of X.

- $g(X) = 0$: $X \cong \mathbb{P}^1$, and

$$\mathrm{End}_X = \coprod_{d \geq 0} \mathrm{End}_X^{(d)},$$

where $\mathrm{End}_X^{(d)}$ parametrizes endomorphisms $\mathbb{P}^1 \to \mathbb{P}^1$ of degree d.

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Example: smooth projective curves

- $g(X) = 1$: $(X, 0) =$ elliptic curve

X acts on itself by translations:

$$X \to \mathrm{Aut}_X, x \mapsto \tau_x \quad \text{where} \quad \tau_x(y) = x + y.$$

Every $g \in \mathrm{End}(X)$ can be written uniquely as $g = \tau_x \circ f$, with $f \in \mathrm{End}_{X,0}$ (i.e. $f \in \mathrm{End}(X)$ and f fixes the origine 0).

We thus obtain

$$X \rtimes \mathrm{End}_{X,0} \xrightarrow{\sim} \mathrm{End}_X, \qquad (x, f) \mapsto \tau_x \circ f$$

where $\mathrm{End}_{X,0}$ acts on X by evaluation.

We have $\pi_0 \mathrm{End}_X \cong \mathrm{End}_{X,0}$ and a commutative diagram

$$\begin{tikzcd} \mathrm{End}_X \arrow[dr, bend left, "q"] & \\ X \times_k \mathrm{End}_{X,0} \arrow[u, "\cong"] \arrow[r, "\mathrm{pr}_2"'] & \mathrm{End}_{X,0} \end{tikzcd} \quad \text{where} \quad q(g) = \tau_{-g(0)} \circ g$$

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- $g(X) \geq 2$: We have

$$\mathrm{End}_X \cong X \coprod \mathrm{Aut}_X \coprod C,$$

where X represents the constant endomorphisms, Aut_X is finite, and C is discrete infinite not appearing in characteristic zero.

Corollary

X is a smooth projective curve $\Longrightarrow$ End_X is smooth.

For X arbitrary, Aut_X is reduced in characteristic zero, but End_X is not reduced for certain surfaces in arbitrary characteristic.

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Finiteness Aspects

Definition

A group G is bounded if the orders of all its finite subgroups are uniformly bounded. We set

$$b_G = \sup\{|F|, F \text{ is a finite subgroup of } G\}.$$

Example (Minkowski, 1887)

$\mathrm{GL}_n(\mathbb{Z})$ is bounded, and

$$b_{\mathrm{GL}_n(\mathbb{Z})} \leq b_{\mathrm{GL}_n(\mathbb{F}_3)} = |\mathrm{GL}_n(\mathbb{F}_3)| = \prod_{i=0}^{n-1}(3^n - 3^i).$$

Example

$\mathrm{GL}_n(\mathbb{Q})$ is bounded, and $b_{\mathrm{GL}_n(\mathbb{Q})} = b_{\mathrm{GL}_n(\mathbb{Z})}$.

Counterexample

$\mathrm{GL}_n(\mathbb{C})$ is not bounded.

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A remedy?

II. Jordan's Property

Theorem (Jordan, 1878)

There exists $d = d(n) \in \mathbb{N}^*$ such that every finite subgroup F of $\mathrm{GL}_n(\mathbb{C})$ contains an abelian normal subgroup A with index $|F : A| \leq d$.

Definition (Popov, 2011)

A group G is Jordan if there exists $d = d(G) \in \mathbb{N}^*$ such that every finite subgroup F of G has an abelian normal subgroup A with $|F : A| \leq d$.

Examples (suppose $\mathrm{char}(k) = 0$ for simplicity)

- Every bounded group is Jordan
- Finite groups, abelian groups
- $\mathrm{GL}_n(k)$
- $\mathrm{Bir}(\mathbb{P}^2_k)$ (Serre, 2008)
- $\mathrm{Aut}(X)$ (Meng and Zhang, 2018)

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What can one say about $\mathrm{End}(X)$?

Subgroups of a Monoid

Definition

Let M be a monoid. A subset G of M is called a subgroup of M if G is a group under the operation of M.

The neutral element e of G is idempotent; it satisfies $e^2 = e$.

Exemple

Let $e \in M$ be an idempotent. The set

$$eMe := \{eme \mid m \in M\} = \{m' \in M \mid m'e = m' = em'\}$$

is a monoid with neutral element e.
$H_e := (eMe)^\times$ is a subgroup of M with neutral element e.

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Subgroups of a Monoid

$$eMe := \{eme \mid m \in M\}, \qquad H_e := (eMe)^\times$$

Example

In $\mathrm{M}_n(k)$, let e be an idempotent of rank r. Then $e = \left(\begin{array}{c|c} \mathrm{id} & 0 \\ \hline 0 & 0 \end{array}\right)$ in an appropriate basis of k^n, where $\mathrm{id} \in \mathrm{M}_r(k)$. We have

$$e\mathrm{M}_n(k)e = \left(\begin{array}{c|c} \mathrm{id} & 0 \\ \hline 0 & 0 \end{array}\right)\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)\left(\begin{array}{c|c} \mathrm{id} & 0 \\ \hline 0 & 0 \end{array}\right) = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 0 \end{array}\right) \cong \mathrm{M}_r(k),$$

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Lemma

- H_e is a maximal subgroup of M (with respect to inclusion).
- H_e is the unique maximal subgroup of M with neutral element e.
- If $e, f \in M$ are distinct idempotents, then H_e and H_f are disjoint.
- Subgroups of M form mutually disjoint lattices, each of which starts with an idempotent and ends with a maximal subgroup.

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First Finiteness Statement

Theorem (End(X) is "locally Jordan")

The maximal subgroups of End(X) are all of the form Aut(Y), with Y the image of X by an idempotent endomorphism. In characteristic zero, all these maximal subgroups are Jordan.

Definition

A monoid M is bounded if the orders of all its finite subgroups are uniformly bounded. We set

$$b_M = \sup\{|F|, F \text{ is a finite subgroup of } M\}.$$

We have

$$b_M = \sup\{b_{H_e} \mid e \in M \text{ is idempotent}\}.$$

Example

The multiplicative monoid $\mathrm{M}_n(k)$ is bounded if and only if the group $\mathrm{GL}_n(k)$ is. We have $b_{\mathrm{M}_n(k)} = b_{\mathrm{GL}_n(k)}$.

Counterexample

The monoid $\operatorname{End}(\mathbb{P}^1)$ is not bounded.

II. Boundedness of $\pi_0 \operatorname{End}_X$

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The monoid $\mathrm{End}(\mathbb{P}^1)$ is not bounded.

II. Boundedness of $\pi_0 \operatorname{End}_X$: action on $\mathsf{N}^1(X)$

The action of endomorphisms on line bundles via pullback

$$(f, \mathscr{L}) \mapsto f^*\mathscr{L}$$

induces an action (via group endomorphisms) of $\pi_0 \operatorname{End}_X$ on the group of Néron-Severi $\mathsf{N}^1(X)$ (line bundles modulo numerical equivalence) ; a free abelian group of finite rank $\rho = \rho(X)$. We call ρ the Picard number of X.

Hence the action of $\pi_0 \operatorname{End}_X$ on $\mathsf{N}^1(X)$ is equivalent to a linear (integral) representaion

$$\beta : \pi_0 \operatorname{End}_X \to \operatorname{End}_{\mathsf{Groups}}(\mathsf{N}^1(X)) \cong \mathsf{M}_\rho(\mathbb{Z}).$$

In particular, we get a homomorphism of groups

$$\beta' : \pi_0 \operatorname{Aut}_X \to \mathsf{GL}_\rho(\mathbb{Z}).$$

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Second Finiteness Statement

Theorem

The fibers of $\beta : \pi_0 \operatorname{End}_X \to \mathrm{M}_\rho(\mathbb{Z})$ are all finite.

Theorem ($\pi_0 \operatorname{End}_X$ is "locally bounded")

- The maximal subgroups of $\pi_0 \operatorname{End}_X$ are all of the form $\pi_0 \operatorname{Aut}_Y$, with Y the image of X by an idempotent endomorphism.
- The Picard numbers of the varieties Y are uniformly bounded by the Picard number of X.
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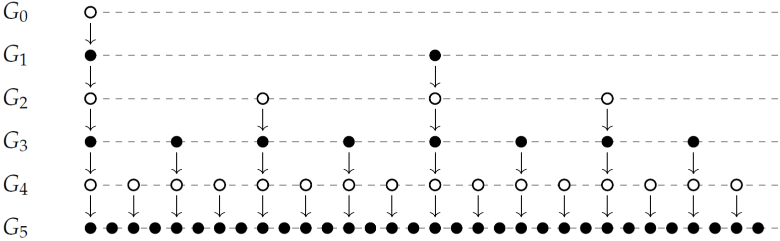
Example

Passing from local to global? Not that simple!

Example



Third Finiteness Statement

Theorem ("global boundedness" of $\pi_0 \operatorname{End}_X$)

Each of the following conditions implies that the monoid $\pi_0 \operatorname{End}_X$ is bounded:

- X is a curve,
- X is a normal surface,
- X is an abelian variety.
- ...

Open Questions

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Is the monoid $\pi_0 \operatorname{End}_X$ bounded for every projective variety X? Or does there exist an example where $\pi_0 \operatorname{End}_X$ is not bounded?

Question

Does there exist a projective variety X admitting infinitely many, pairwise non-isomorphic retracts?

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Thank you for your attention!