DEF. (GRADED POISSON ALGEBRA)

$$(P, \cdot, \{\cdot,\cdot\}) \qquad P = \bigoplus_{m \geq 0} P_m$$

$$P_0 = \mathbb{C} \qquad \dim P_m < \infty$$

- $\{\cdot,\cdot\}$ Lie bracket
- Leibniz rule
  $$\{a, bc\} = \{a,b\}c + \{a,c\}b$$
- (GRADED): $P_m \cdot P_k \subseteq P_{m+k}$
  $$\{P_m, P_k\} \subseteq P_{m+k-1}$$

DEF. (FILTERED ASSOCIATIVE ALG.)

$$(Q, \cdot) \qquad Q = \bigcup_{m \geq 0} F_m Q \qquad \dim F_m Q < \infty$$

$$\mathbb{C} = F_0 Q \subseteq F_1 Q \subseteq \cdots$$

$$F_m Q \cdot F_k Q \subseteq F_{m+k} Q$$

LIE BRACKET

$$[F_m Q, F_k Q] \subseteq F_{m+k-1} Q$$

RMK. $$\mathrm{gr}\, Q := \bigoplus_{m \geq 0} F_m Q / F_{m-1} Q \qquad F_{-1} Q = 0$$

is a GRADED POISSON ALG.

$$(a+F_{n-1})\cdot(b+F_{k-1}) := ab + F_{n+k-1}$$

$$\{a+F_{n-1}, b+F_{k-1}\} := [a,b] + F_{n+k-2}$$

DEF. (FILTERED QUANTIZATION)

P gr. Poisson alg.

(Q, i) . Q filt. ass. alg.

. $i : \operatorname{gr} Q \xrightarrow{\sim} P$

isom. of gr. Poiss. alg.

Ex. (V, ω) sympl. v. sp.

$S(V)$ Poisson alg. $\{u,v\} = \omega(u,v)$ $u,v \in V$

$$W(V) = T(V) / \langle u\otimes v - v\otimes u - \omega(u,v) \mid u,v \in V \rangle$$

is quant. of $S(V)$

EG. $\dim V = 2$ $\mathbb{C}[x,y] = S(V)$

$$W(V) = \mathbb{C}\langle x,y\rangle / (xy - yx = 1)$$

Ex. $S(\mathfrak{g})$, $U(\mathfrak{g})$

Q. How much of Q does P remember?
What about automorphisms?

CONJ. (BELOV-KANEL, KONTSEVICH)

$$\mathrm{Aut}(W(V)) \cong \mathrm{PAut}(S(V))$$

FACT. • True for $\dim V = 2$ (explicit check)

• True for "tame" automorphisms

DEF. (FILTERED DEFORMATION)

(D, i) s.t. • D Filt. Poiss. alg.

$$D = \bigcup_{m \geq 0} D_m \qquad \{D_m, D_k\} \subseteq D_{m+k-1}$$

• $i : \mathrm{gr}\, D \xrightarrow{\sim} P$ iso of graded Poisson alg.

DEF. (SYMPLECTIC QUOTIENT SINGULARITIES)

(V, ω) sympl. v. sp.

$G \leq Sp(V)$ Finite $\qquad G \curvearrowright V$

$$V/G := \mathrm{Spec}(\mathbb{C}[V]^G)$$

Are examples of conical sympl. sing.

(very well-behaved non-smooth varieties)

Thm. (Namikawa + Losev)

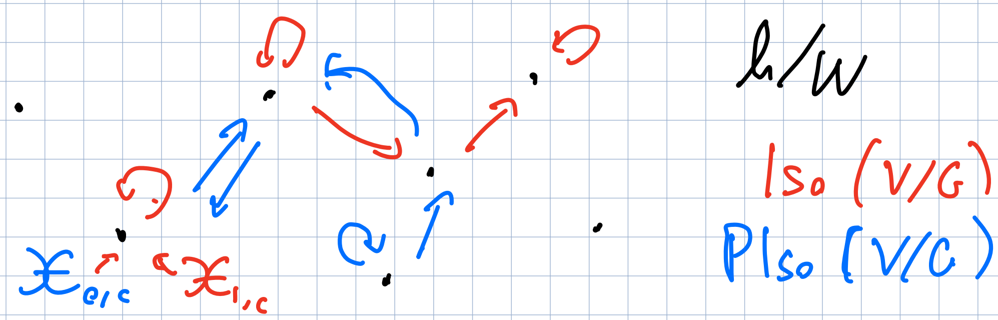
The parameter space of deformations and quantizations of $\mathbb{C}[V/G]$ are the same

$\mathcal{M} = \mathfrak{h}/W$ $\qquad \mathfrak{h} = \{c : S \to \mathbb{C} \mid \text{conj. invariant}\}$

S = set simple refl.

- gen. Cartan space
- W Namikawa Weyl group $W \curvearrowright \mathfrak{h}$ as refl. grp.

$\Rightarrow \mathfrak{h}/W$ is affine space



Cons. $\mathrm{Iso}(V/G) \simeq \mathrm{PIso}(V/G)$ as groupoids

Rmk.

- $G = \langle id \rangle \rightarrow V/G = V$ $\qquad \mathfrak{h}/W = pt.$

  $\rightarrow$ BKK-Cons. $(\mathfrak{h} = 0)$

- Holds for filtered isomorphisms (by universal properties)

($\exists$ univ. def + univ. quant.,
univ. quant $\xrightarrow{\simeq}$ univ. def.)

. What about $n=2$?

FACT. Sympl. quot. sing. in dim $=2$
are exactly the Kleinian singularities
$G \leq Sp(2) = SL(2)$
have ADE classifications.

$$\underset{n\geq 1}{A_n}, \quad \underset{n\geq 4}{D_n}, \quad E_6, E_7, E_8$$

V/G is a surf. in $\mathbb{C}^3$

EG. $\mathbb{C}[A_{n-1}] = \mathbb{C}[x,y,z]/(xy = z^n)$

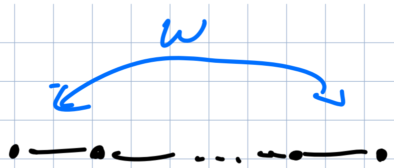
$\mathbb{C}[D_n] = \mathbb{C}[x,y,z]/(x^{n-1} + xy^2 + z^2 = 0)$

THM. (C.)

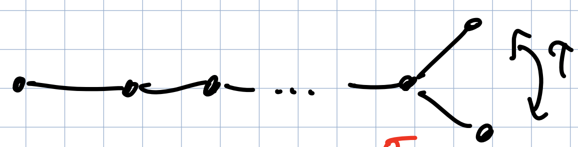
$$\mathrm{Iso}(V/G) \cong \mathrm{PIso}(V/G)$$

where $V/G = A_n$ or D_n

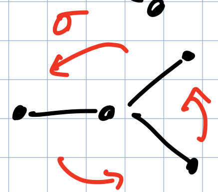
(so Conj. is confirmed in dim $=2$
except type E)



A_n



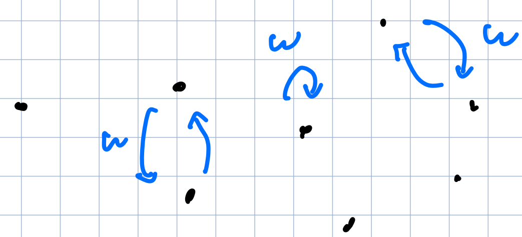
D_n



D_4

$$\mathrm{Aut}(\Gamma) = \begin{cases} \mathbb{Z}_2 & A_n \\ \mathbb{Z}_2 & D_n \quad n>4 \\ S_3 & D_4 \end{cases}$$

$\mathrm{Aut}(\Gamma) \curvearrowright \mathfrak{h}/W$ inducing filt. iso.



A_n

$\mathrm{Iso}(V/G)$ is generated by $\mathrm{Aut}(\Gamma)$
+ automorphisms

Type A $\quad \mathfrak{h}/W \quad$ actual Cartan and Weyl grp. of corresponding Lie alg.

$\Rightarrow \mathfrak{h}/W = \mathbb{C}^{m-1}$

$\rightsquigarrow$ $P(z)$ of deg m, monic, no term of deg $m-1$

$= z^m + a_1 z^{m-2} + \dots + a_m$

$$\mathbb{C}[x,y,z]/(xy - P(z))$$

$H_\lambda : (x,y,z) \mapsto (\lambda x, \lambda^{-1} y, z) \qquad \lambda \in \mathbb{C}^*$

$\exp(\mathrm{ad}(g(x))) \,/\, \exp(\mathrm{ad}(g(y)))$

TRIANGULAR AUT.

THESE ARE NOT FILTERED

For special par. (when $P(z)$ is even/odd)

$\omega : (x,y,z) \mapsto (y, (-1)^m x, -z)$

TYPE D:

$\mathfrak{h}/W = \mathbb{C}^m$

$\rightsquigarrow Q(x) = x^{m-1} + a_1 x^{m-1} + \dots + a_{m-1}$

$\gamma \in \mathbb{C}$

$$\mathbb{C}[x,y,z]/(Q(x) + xy^2 + z^2 - \gamma y)$$

$\sigma : (x,y,z) \mapsto (x,-y,-z)$

FOR SPECIAL PAR. $(\gamma = 0)$

τ : CONDITION ON Q $\qquad (m=4)$

$+ \ \gamma = 0$

$\sigma \circ \tau$: $\gamma \neq 0$, CONDITION ON Q $(n=4)$

$$\Rightarrow \quad \mathrm{Aut} = \begin{cases} \mathrm{id} & (\text{generic}) \\ \mathbb{Z}_2 & (\gamma = 0,\ n \geq 4) \\ S_3 & (n=4,\ \gamma=0,\ Q \text{ nice}) \\ \mathbb{Z}_2 & (n=4,\ \gamma\neq 0,\ Q \text{ nice}) \\ \mathbb{Z}_2 & (n=4,\ \gamma=0,\ Q \text{ not nice}) \end{cases}$$