# Cluster algebras from the perspective of operator algebras

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# Positive representations

**Goal:** Find a good framework for positive representations

- Pos. rep. ≈ Represent quantum cluster variables by positive operators
- Also want non-faithful representations
- We have a working definition

# Positive characters

- Simplest setting:
  - $\mathcal{H} = \mathbb{C}$
  - $A$ a *commutative* cluster algebra (of geometric type)

A positive character is a unital algebra homomorphism $\chi : A \to \mathbb{C}$ such that $\chi(a) \geq 0$ for all cluster variables $a$ in $A$.

- $\mathrm{Spec}(A)$: All characters
- $\mathrm{Spec}_{\geq 0}(A)$: All positive characters
- $\mathrm{Spec}_{>0}(A)$: All strictly positive characters

**Warning:** $A$ has more relations than just exchange relations

# Example: $G = \mathrm{SL}(n, \mathbb{C})$

- $X \in G = \mathrm{SL}(n, \mathbb{C})$ is *totally positive* if $\Delta_{I,J}(X) \geq 0$ for all minors $\Delta_{I,J}$
- Gives $G_{\geq 0}$ (and analogously $G_{>0}$)
- There exists a cluster structure $A$ on $\mathbb{C}[G]$ (see CA book chapter 6, [FWZ21])

$\rightsquigarrow$ Canonical identifications $\mathrm{Spec}_{\geq 0}(A) = G_{\geq 0}$ and $\mathrm{Spec}_{>0}(A) = G_{>0}$

- Also possible for other $G$ (partial compactification of cluster structure by BFZ, see [Oya25])

- Equip $\mathrm{Spec}(A)$ with Euclidean topology

  $\rightsquigarrow$ $\mathrm{Spec}_{\geq 0}(A)$ is closed in $\mathrm{Spec}(A)$

Questions/Conjectures:

- Is $\mathrm{cl}(\mathrm{Spec}_{>0}(A)) = \mathrm{Spec}_{\geq 0}(A)$? (Density property)
- Is $\mathrm{int}(\mathrm{Spec}_{\geq 0}(A)) = \mathrm{Spec}_{>0}(A)$? (Interior property)

## Different generators

- Let $U_{\geq 0} = \bigcap_{\boldsymbol{x}} \mathbb{Z}_{\geq 0}[\boldsymbol{x}^{\pm 1}]$ be the universally positive Laurent functions
- Every cluster variable is contained in $U_{\geq 0}$ (positive Laurent phenomenon)
- Define $\widetilde{\mathrm{Spec}}_{\geq 0}(A)$ w.r.t $A \cap U_{\geq 0}$

Question: Is $\widetilde{\mathrm{Spec}}_{\geq 0}(A) = \mathrm{Spec}_{\geq 0}(A)$?

If density property holds, then yes

# Cluster localizations

- Let $\tilde{A}$ be obtained from $A$ by freezing (and inverting) some cluster variables in a seed
- If $A \subseteq \tilde{A}$ we call it a *cluster localization* (introduced by G. Muller [Mul13])
- Let $A_+$ and $\tilde{A}_+$ be the semirings generated by the cluster variables
- Is it then also a *positive* cluster localization $A_+ \subseteq \tilde{A}_+$?

  $\rightsquigarrow \mathrm{Spec}_{\geq 0}(\tilde{A}) \to \mathrm{Spec}_{\geq 0}(A)$
- Remark: positive Laurent phenomenon is a special case!

# Algebraic properties

- Does $\mathrm{Spec}_{\geq 0}(A)$ tell us something about $A$?
- $A$ finitely generated $\Rightarrow$ $\mathrm{Spec}_{\geq 0}(A)$ is locally compact
- $A$ locally acyclic $\Rightarrow$ $\mathrm{Spec}_{\geq 0}(A) = \mathrm{Spec}_{>0}(A)$
- Are the converse statements true?

Final (seemingly unrelated) question:

- Let $A_1, A_2, A_3, \ldots$ be successive clusters in a quantum cluster algebra $A$
- Let $[CD] = \text{span}\{cd \mid c \in C, d \in D\} \subseteq A$
- Is it true that $[A_1 A_2 \cdots A_n] = [A_n A_1 A_2 \cdots A_{n-1}]$ for all $n$?

[FWZ21] S. Fomin, L. Williams, and A. Zelevinsky, "Introduction to Cluster Algebras. Chapter 6." [Online]. Available: https://arxiv.org/abs/2008.09189

[Oya25] H. Oya, "A note on a cluster structure of the coordinate ring of a simple algebraic group." [Online]. Available: https://arxiv.org/abs/2504.09011

[Mul13] G. Muller, "Locally acyclic cluster algebras," *Advances in Mathematics*, vol. 233, no. 1, pp. 207–247, Jan. 2013, doi: 10.1016/j.aim.2012.10.002.